

Age Based Preferences in Paired Kidney Exchange*

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Abstract

We consider a model of Paired Kidney Exchange (PKE) with feasibility constraints on the number of patient-donor pairs involved in exchanges. Patients' preferences are restricted so that patients prefer kidneys from compatible younger donors to kidneys from older donors. In this framework, patients with compatible donors may enroll on PKE programs to receive an organ with higher expected graft survival than that of their intended donor. PKE rules that satisfy individual rationality, efficiency, and strategy-proofness necessarily select pairwise exchanges. Such rules maximize the number of transplantations among pairs with the youngest donors, and sequentially among pairs with donors of different age groups. JEL: C78; D02; D78; I10.

Keywords: Kidney exchange; age based preferences, Priority rules, Strategy-proofness.

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1 Introduction

Paired Kidney Exchange (PKE) programs seek to overcome any incompatibility (of blood or tissue types)¹ of living donor-patient pairs by arranging swaps of donors among several pairs [8, 9, 30]. PKE programs work as clearing houses that periodically search for mutually compatible exchanges of donors in a pool of donor-patient pairs. In order to find such mutually compatible exchanges, PKE programs need to elicit relevant information from patients (and their doctors) and to overcome feasibility constraints that are absent in standard problems of allocation of indivisible goods. Specifically, PKE programs involve the cooperation and coordination of several transplantation units at different medical centers. Thus, the complexity of the logistics makes exchanges involving too many donor-patient-pairs unfeasible.² For this reason, real-life PKE programs have generally focused on maximizing the number of simultaneous compatible organ exchanges between two donor-patient pairs, although swaps involving more than two pairs are also carried out. To deal with situations where a donor-patient pair may be necessary in more than one compatible exchange of donors, real-life PKE programs usually give priority to particular patients in much the same way as happens in the allocation of kidneys obtained from cadaveric donors.³

Living-donor kidney transplantation yields excellent results in terms of life expectancy of the graft compared to kidney transplantation from cadaveric organs [8, 13]. This fact explains the prevalent approach in PKE programs, which assumes that patients only care about receiving a compatible kidney. Recent medical research, however, supports the idea that different compatible kidneys can have substantially different outcomes. The age and the health status of the donor, in fact, have a major impact on the expected survival of

¹A patient and a donor are incompatible if the patient body will immediately reject the donor's kidney after the graft, and thus the transplantation is deemed not viable.

²There is growing interest in the creation of non-simultaneous, extended altruistic-donor chains that try to avoid such limitations. A donor chain starts with an altruistic donor willing to donate to anyone needing a kidney transplant without having a related recipient [4, 28, 40].

³This is the case of the New England PKE program [31, 32]. Similar protocols are adopted by other centralized PKE programs implemented in countries like Korea, the Netherlands, the United Kingdom, Spain, and the United States with its UNOS National Pilot Program for Kidney Paired Donation [2, 3, 17, 20, 25, 27].

the graft [11, 12, 22, 38].⁴ This observation has important implications for modeling PKE. First, the heterogeneity of transplantation outcomes may affect participants’ incentives. Secondly, it provides a justification for the participation in PKE programs of patients with a compatible willing donor. These pairs may have incentives to participate in PKE programs since the patient could obtain a kidney that results in higher life-expectancy than that of her donor’s kidney. The participation of compatible pairs can dramatically increase the chances of finding compatible swaps for incompatible pairs, and boost the transplantation rate [11, 30, 32, 33].

We model PKE clearing houses as rules that assign the donors’ kidneys to the patients taking into account patients’ preferences over available kidneys. Each patient’s preferences depend on the donors’ characteristics that determine compatibility as well as life expectancy after transplantation. The pool of available kidneys can be partitioned into groups of kidneys of similar quality corresponding to the donors’ age groups. Patients are interested in receiving a compatible kidney, but if possible they prefer a kidney from a younger donor. This observation suggests the analysis of a restricted domain of preferences: the *age based preference domain*. In this domain, we study rules that satisfy individual rationality,⁵ efficiency restricted by the logistic constraints, and strategy-proofness.⁶ Those properties are incompatible for rules that allow for simultaneous swaps involving more than two donor-patient pairs. Hence, we focus on rules that only allow for pairwise exchanges among two donor-patient pairs. We propose a family of rules – *age based priority rules* – that adapt the PKE protocols used in practice to the age based preference domain. According to age based priority rules, patients sequentially select the pairwise assignments they prefer from the set of individually rational pairwise assignments, and patients’ positions in the sequential choice procedure are determined by the age of their donors. If patients can strictly rank all compatible kidneys, then age

⁴For instance, Øien et al. [22] confirm that the donor’s age and health status have a crucial role in the case of living donations. A donor over 65 years old is associated with a higher risk of graft loss at every time point after transplantation. There is more controversy in the medical literature regarding the effects of other characteristics such as the similarity of tissue types between patients and donors [8, 13, 23, 24].

⁵A rule satisfies individual rationality if patients never prefer the initial assignment where no kidney swap is performed to the outcome prescribed by the rule.

⁶A rule satisfies strategy-proofness if patients never have incentives to misrepresent their preferences.

based priority rules are characterized by our three natural properties. If patients may be indifferent among compatible kidneys, the rules that satisfy the above properties and the auxiliary property of non-bossiness select assignments that maximize the number of exchanges among pairs with the youngest donors and sequentially among pairs in different age groups according to a priority ordering based on donors' age.⁷

Our positive results rely on the analysis of the existence of strict core assignments in the non-monetary exchange problem defined by PKE. A kidney assignment is in the strict core if no group of patients can (weakly) benefit by swapping donors among themselves. Under strict preferences and no feasibility constraints, the strict core is single-valued and the rule that selects the unique core allocation satisfies strategy-proofness and many other desirable properties [5, 18, 19, 29, 35]. This is not the case when indifferences are allowed and/or there are feasibility constraints [1, 10, 16]. For general assignments problems with non-empty strict core, Sönmez [36] shows that there are rules that satisfy individual rationality, (constrained) efficiency, and strategy-proofness only if the strict core is unique. That result is crucial for our analysis under strict preferences and pairwise exchanges, since in this case the strict core coincides with the assignment selected by an age based priority rule. In the general case with indifferences, the strict core may be empty, but under our domain restriction, we show that there are rules that satisfy those desirable properties, and such rules are naturally related to age based priority rules.

This paper contributes to the kidney exchange literature initiated by Roth et al. [30]. In recent years, PKE has received a considerable interest from both a theoretical and a practical design point of view. Most works have considered the framework that incorporates specific features consistent with the medical approach to PKE in New England [14, 31, 32, 34, 40–42]. That approach assumes that only incompatible pairs participate in PKE, and that patients are indifferent between two compatible kidneys. In this paper, we introduce a restricted domain of preferences, which allows for strict preferences over compatible kidneys and provides incentives for compatible pairs to enroll on PKE programs, but inevitably introduces additional structure for satisfactory rules. Finally, we refer to two related recent papers. Sönmez and Ünver [37] analyze the structure of Pareto

⁷A rule satisfies non-bossiness if, whenever a change in a patient's preferences does not affect the kidney she receives, it does not modify any other patient's assignment.

efficient matchings when compatible pairs are admitted into PKE in the New England PKE framework. In that paper, the sole motivation for the participation of compatible pairs is altruism. Nicolò and Rodríguez-Álvarez [21] propose a model where patients' preferences are unrestricted but the only private information is the minimal quality of the kidney that each patient requires to undergo transplantation. In that framework, under any arbitrary restriction on the number of pairs involved in exchanges, no rule satisfies individual rationality, (constrained) efficiency, and strategy-proofness.⁸

Before proceeding with the formal analysis, we briefly outline the contents of the remainder of this paper. In Section 2, we present basic notation and definitions. In Section 3, we introduce the concept of age based preferences. In Section 4, we state our results. In Section 5, we conclude. We provide all the proofs in the Appendix.

2 Basic Notation

Consider a finite society consisting of a set $N = \{1, \dots, n\}$ of patients ($n \geq 3$) who need a kidney for transplantation. Each patient has a potential donor, and $\Omega = \{\omega_1, \dots, \omega_n\}$ denotes the set of kidneys available for transplantation. For each patient i , ω_i refers to the kidney of patient i 's donor. We assume that all available kidneys are obtained from living donors and each patient has only one potential donor.

Each patient i is equipped with a complete, reflexive, and transitive preference relation \succsim_i on Ω . We denote by \succ_i the associated strict preference relation and by \sim_i the associated indifference relation. Let \mathcal{P} denote the set of all preferences. We call $\succsim = (\succsim_i)_{i \in N} \in \mathcal{P}^n$ a preference profile. For each patient i and each $\succsim \in \mathcal{P}^N$, $\succsim_{-i} \in \mathcal{P}^{n-1}$ denotes the restriction of the profile \succsim for all the patients excluding i . We assume that patients' preferences are further restricted, so that for each patient i her preferences belong to a subset $\mathcal{D}_i \subset \mathcal{P}$. We denote by $\mathcal{D} \equiv \times_{i \in N} \mathcal{D}_i \subseteq \mathcal{P}^N$ a domain of preference profiles over kidneys.

⁸Zenios [43] also considers PKE where patients care about the quality of the outcome in a dynamic setting but there is no information to be elicited from the patients. The focus is on the optimal assignment of donor-patient pairs to direct exchange programs or indirect exchange programs, where patients may swap their incompatible donor to gain priority on the waiting list.

An **assignment** a is a bijection from kidneys to patients. For each patient i and each assignment a , a_i denotes the kidney assigned to i by a . Whenever $a_i = \omega_i$, we consider that either patient i continues in dialysis or – if she is compatible with her donor – she receives her donor’s kidney. Let \mathcal{A} be the set of all assignments.

Kidneys are assigned to patients by forming cycles of patient–donor pairs. In each cycle, every patient receives a kidney from the donor of some patient in the cycle, and her donor’s kidney is transplanted to another patient in the cycle. For each assignment a , let π_a be the finest partition of the set of patients such that for each $p \in \pi_a$ and each $i \in p$, there are $j, j' \in p$, with $a_i = \omega_j$ and $a_{j'} = \omega_i$.⁹ For each assignment a the partition π_a is unique and well-defined. We define **cardinality of** a as the $\max_{p \in \pi_a} \#p$.¹⁰ The cardinality of an assignment refers to the size of the largest cycle formed in the assignment.

For each $k \in \mathbb{N}$, $k \leq n$, we say that the assignment a is **k -feasible** if a ’s cardinality is not larger than k . Let \mathcal{A}^k be the set of all k -feasible assignments.

Given a preference domain \mathcal{D} , a rule is a mapping $\varphi : \mathcal{D} \rightarrow \mathcal{A}$. Since we consider feasibility constraints, for some $k < n$, we focus on rules that only select k -feasible assignments, $\varphi : \mathcal{D} \rightarrow \mathcal{A}^k$.

The most binding feasibility restriction appears when only exchanges between two donor-patient pairs are admitted. An assignment a is a **pairwise assignment** if $a \in \mathcal{A}^2$. A rule φ is a **pairwise exchange rule** if $\varphi : \mathcal{D} \rightarrow \mathcal{A}^2$.

We are interested in rules that satisfy the following standard properties.

Individual Rationality. For each $i \in N$ and each $\succsim \in \mathcal{D}$, $\varphi_i(\succsim) \succsim_i \omega_i$.

k -Efficiency. For each $\succsim \in \mathcal{D}$, there is no $a \in \mathcal{A}^k$ such that for each $i \in N$ $a_i \succsim_i \varphi_i(\succsim)$ and for some $j \in N$, $a_j \succ_j \varphi_j(\succsim)$.

Note that for a rule $\varphi : \mathcal{D} \rightarrow \mathcal{A}^k$, *k-efficiency* refers to efficiency restricted to the set of feasible assignments.

⁹Note that $i = j = j'$ and $a_i = \omega_i$ are allowed.

¹⁰For each set S , $\#S$ refers to the number of elements of S . Thus, for each element p of the partition π_a , $\#p$ refers to the number of patients involved in the exchange cycle.

Strategy-Proofness. For each $i \in N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_i \in \mathcal{D}_i$, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_i, \succsim_{-i})$.

In addition to these basic properties, we consider one additional desirable property.

Non-bossiness. For each $i \in N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_i \in \mathcal{D}_i$, $\varphi_i(\succsim) = \varphi_i(\succsim'_i, \succsim_{-i})$ implies $\varphi(\succsim) = \varphi(\succsim'_i, \succsim_{-i})$.

Non-Bossiness is a standard technical property in environments that admit indifference. Since monetary transactions related to organ donation are almost universally banned, *non-bossiness* has also a reasonable normative justification in PKE problems. A rule that violates *non-bossiness* may give rise to illegal bribes among donor-patient pairs. If a patient i changes her preference report and affects the outcome of patient j , then i may have incentives to accept a monetary compensation from j in order to reverse her report.

3 Age Based Preferences

In this section we present a new domain restriction that is directly inspired by the specifics of PKE. We start by introducing some useful notation.

For each patient i and each preference $\succsim_i \in \mathcal{P}$, let $D(\succsim_i) \equiv \{\omega \in \Omega \setminus \{\omega_i\} \mid \omega \succ_i \omega_i\}$ define the **set of desirable kidneys for patient i** . Alternatively, let $ND(\succsim_i) \equiv \{\omega \in \Omega \setminus \{\omega_i\} \mid \omega_i \succsim_i \omega\}$ define the **set of undesirable kidneys for patient i** .

The set $D(\succsim_i)$ contains all the kidneys which lead to an improvement with respect to i 's outside option ω_i : that is, either staying in dialysis or receiving her donor's kidney. Conversely, $ND(\succsim_i)$ contains all incompatible kidneys that lead to i 's rejection of the graft as well as those kidneys that may lead to a poor transplantation outcome, and i prefers to keep ω_i rather than receive them.

The viability (non-rejection) of transplantation depends on the compatibility of tissue and blood types that are idiosyncratic to each patient and donor. There are a series of characteristics of the donors that are crucial in determining the final outcome in terms of expected life-expectancy after transplantation. Specifically, donor's age and health status turn out to be the most important characteristics in determining the probability

of long-term graft survival in the case of living donations. Both characteristics are quite closely correlated and directly observable by transplant coordinators, and they affect all the patients in the same way. The age of the donors does not determine the possibility of rejection of a kidney by a patient. However, whenever a patient compares two desirable kidneys, she prefers the kidney from the youngest donor. If the donors of both kidneys are (approximately) of the same age, then she is indifferent between them. We therefore assume that there is a characteristic, namely the age of the donor, according to which available kidneys can be partitioned into groups of kidneys of the same quality. The following notation formalizes this idea.

An **age structure** is a partition $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ of Ω . We consider that for each $\omega, \omega' \in \Omega$ and $l, l' \in \mathbb{N}$, $\omega \in \Pi(l)$ and $\omega' \in \Pi(l')$ $l \leq l'$ imply that the donor of ω is not older than the donor of ω' . Thus, we call each element of the age structure an **age group**.

Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be an age structure. For each patient $i \in N$, the preference relation $\succsim_i \in \mathcal{P}$ is a Π **age based preference** if for each $\omega, \omega' \in D(\succsim_i)$ and for each $\bar{\omega} \in ND(\succsim_i)$:

- (i) $\omega \in \Pi(j)$ and $\omega' \in \Pi(k)$ and $j < k$ imply $\omega \succ_i \omega'$,
- (ii) $\omega, \omega' \in \Pi(j)$ implies $\omega \sim_i \omega'$, and
- (iii) $\omega_i \succ_i \bar{\omega}$.

Let \mathcal{D}_i^Π denote the set of all Π age based preferences for patient i and let $\mathcal{D}^\Pi \equiv \times_{i \in N} \mathcal{D}_i^\Pi$ denote the domain of Π age based preferences.

Without any loss of generality and to simplify notation, we henceforth assume that for each age structure Π , for each $i, j \in N$, if $i < j$, $\omega_i \in \Pi(l)$, and $\omega_j \in \Pi(l')$, then $l \leq l'$.

We denote with Π^* the **natural age structure** such that $\Pi^* = \{\Pi^*(1), \dots, \Pi^*(n)\}$ and for each $l \leq n$, $\Pi^*(l) = \{\omega_l\}$. According to Π^* , kidneys are strictly ordered according to the natural order that corresponds to donors' ages. Of course, according to Π^* age based preferences, patients are never indifferent between two desirable kidneys. Alternatively, let $\bar{\Pi}$ denote the coarsest partition such that $\bar{\Pi} = \{\bar{\Pi}(1)\}$ with $\bar{\Pi}(1) = \Omega$. The domain

$\mathcal{D}^{\bar{\Pi}}$ corresponds to the domain of dichotomous preferences introduced by Roth et al. [31, 32]. According to $\bar{\Pi}$ age based preferences, patients are always indifferent between two desirable kidneys.

Throughout the paper we consider kidney exchange problems such that the age structure Π is sufficiently rich. That is, we consider partitions with at least two age groups and we rule out the case in which the set of kidneys is partitioned in two sets and one of them contains a unique element.¹¹

Assumption A. The age structure Π is such that either $\#\Pi \geq 3$ or $\Pi = \{\Pi(1), \Pi(2)\}$ and $\#\Pi(1) \geq 2$ and $\#\Pi(2) \geq 2$.

4 Age Based Preferences and Priority Rules

4.1 A Preliminary Impossibility Result

In the general model of exchange of indivisible goods without feasibility restrictions, it is well-known that it is possible to design rules that satisfy *individual rationality*, *(n)-efficiency*, and *strategy-proofness*. For instance, when all agents are endowed with strict preferences, the rule that selects Shapley and Scarf’s Top-Trading Cycle (TTC) algorithm outcome for each preference profile satisfies these properties [35].¹² However, according to Theorem 1 in Nicolò and Rodríguez-Álvarez [21] these properties are no longer compatible when there are feasibility constraints on the set of available allocations and the preference domain is unrestricted. For age based domains, since donors’ characteristics have a common effect on patients’ preferences, this negative result does not apply. Nevertheless, our first result highlights the tension among *individual rationality*, *k-efficiency*, and *strategy-proofness* if assignments involving at least three donor-patient pairs are admitted.

¹¹Assumption A is very natural in the PKE framework. It only plays a role in the proof of Theorem 1, and it does not affect the remaining results where we focus on pairwise exchange rules.

¹²See Alcalde-Unzu and Molis [1], Jaramillo and Manjunath [16] for adaptations of the TTC that maintain these properties in preference domains that admit indifferences.

Theorem 1. *For each age structure Π and each $k \in \mathbb{N}$ such that $3 \leq k \leq n - 1$, no rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, and strategy-proofness.*

The proof of Theorem 1 replicates the argument of proof of Theorem 1 in Nicolò and Rodríguez-Álvarez [21]. For each Π that satisfies Assumption A, the preference profiles used in the proof of that result belong to \mathcal{D}^Π . Because Nicolò and Rodríguez-Álvarez [21] study rules defined on the unrestricted domain of preferences, both theorems are logically independent.¹³ Theorem 1 has no implication for $k = 2$ and in what follows we focus on pairwise exchange rules.

4.2 Age Based Priority Rules

The literature on PKE has focused on priority mechanisms that resemble the protocols commonly used to allocate cadaveric organs [31, 32]. In this section, we provide specific definitions to tailor priority mechanisms to PKE problems when patients are equipped with age based preferences.

A **priority ordering** σ is a permutation of patients ($\sigma : N \rightarrow N$) such that the k -th patient in the permutation is the patient with the k -th priority. Let σ^* denote the **natural priority ordering** such that for each $i \in N$, $\sigma^*(i) = i$. For each age structure Π and each priority ordering σ , we say that σ **respects** Π if for every $i, j \in N$, $\omega_i \in \Pi(l)$, $\omega_j \in \Pi(l')$, and $l < l'$ imply $\sigma(i) < \sigma(j)$.

For each $\succsim \in \mathcal{P}^n$, let $\mathcal{I}(\succsim) \equiv \{a \in \mathcal{A}^2 \mid \text{for each } i \in N, a_i \succsim_i \omega_i\}$ denote the **set of all individually rational pairwise assignments**.

Priority Algorithm. Fix a permutation of the patients σ , and a preference profile $\succsim \in \mathcal{P}^N$:

- Let $\mathcal{M}_0^\sigma(\succsim) = \mathcal{I}(\succsim)$.
- For each $t \leq n$, let $\mathcal{M}_t^\sigma \subseteq \mathcal{M}_{t-1}^\sigma$ be such that:

$$\mathcal{M}_t^\sigma(\succsim) = \{a \in \mathcal{M}_{t-1}^\sigma \mid \text{for no } b \in \mathcal{M}_{t-1}^\sigma(\succsim), b_{\sigma^{-1}(t)} \succ_{\sigma^{-1}(t)} a_{\sigma^{-1}(t)}\}.$$

¹³If Assumption A is dropped, we can provide a similar impossibility result for $3 \leq k \leq n - 2$.

Note that \mathcal{M}_n^σ is well defined, non-empty, and essentially single-valued.¹⁴

A pairwise exchange rule $\varphi : \mathcal{D} \rightarrow \mathcal{A}^2$ is a **priority rule** if there is a priority ordering σ such that for each $\succsim \in \mathcal{D}$, $\varphi(\succsim) \in \mathcal{M}_n^\sigma(\succsim)$. We denote by ψ^σ a **priority rule with priority ordering** σ . Given an age structure Π , a pairwise priority rule ψ^σ is an **age based priority rule** if σ respects Π .¹⁵

A priority rule proceeds as a serial dictatorship rule in which patients sequentially select their preferred allocations in $\mathcal{I}(\succsim)$.¹⁶ Age based priority rules assign priority to patients based on the age of their donors. Since all patients prefer younger desirable donors to older ones, age based priority rules are normatively appealing because they give priorities to those patients who bring the most valuable endowments to the PKE program. Moreover, age based priority rules provide incentives to patients (and to the transplant centres with which they are enrolled) to participate in centralized PKE programs instead of looking for alternative private arrangements among groups of donor-patient pairs. To show these features of the age based priority rules, we introduce the concept of the core in pairwise PKE problems. Note that we deal with pairwise exchange rules, so that and without loss of generality, we only need to consider arrangements involving up to two donor-patient pairs.

For each pair $a, b \in \mathcal{A}^2$, each pair $\{i, j\} \subset N$, and each $\succsim \in \mathcal{P}^N$, $\{i, j\}$ **weakly blocks** a at \succsim via b if

- (i) $b_i = \omega_j$ and $b_j = \omega_i$,

¹⁴A set is essentially single-valued either if it is single-valued or if it contains more than one element and all the patients are indifferent between any two elements in the set. That is, for each patient i , each $\succsim \in \mathcal{P}^N$, and each $a, a' \in \mathcal{M}_n^\sigma(\succsim)$, $a_i \sim_i a'_i$.

¹⁵Note that a priority rule may be defined by different priority orderings. In fact, whenever the last element of Π is a singleton, there are priority orderings σ, σ' such that σ respects Π , σ' does not respect Π , and $\psi^\sigma \equiv \psi^{\sigma'}$. See Remark 1 and Lemma 2 in the Appendix.

¹⁶In our pairwise PKE framework, a *serial dictatorship* (for a priority ordering σ) would be defined with the same algorithm but starting from the set \mathcal{A}^2 . Serial dictatorships play a central role in house allocation problems, in which a set of objects has to be assigned to a set of agents but there are neither property rights over the objects nor monetary transfers [15, 26, 39]. Bogomolnaia and Moulin [6] consider a problem of random house allocation under common preferences over houses similar to age based preferences. In that paper, a random version of serial dictatorship is termed a random priority mechanism.

(ii) $b_i \succsim_i a_i$ and $b_j \succsim_j a_j$, and either $b_i \succ_j a_i$ or $b_j \succ_j a_j$ (or both).

Alternatively, $\{i, j\}$ **strongly blocks** a at \succsim via b if

(i) $b_i = \omega_j$ and $b_j = \omega_i$,

(ii) $b_i \succ_i a_i$ and $b_j \succ_j a_j$.

For each $\succsim \in \mathcal{P}^n$, an assignment $a \in \mathcal{A}^2$ is in the **strict core of the pairwise exchange problem associated with** $\succsim -a \in \mathcal{C}(\succsim)$ – if $a \in \mathcal{I}(\succsim)$ and there are no pair $\{i, j\} \subseteq N$ and assignment $b \in \mathcal{A}^2$ such that $\{i, j\}$ weakly blocks a at \succsim via b . Finally, an assignment $a \in \mathcal{A}^2$ is in the **weak core of the pairwise exchange problem associated with** $\succsim -a \in \tilde{\mathcal{C}}(\succsim)$ – if $a \in \mathcal{I}(\succsim)$ and there are no pair $\{i, j\} \subseteq N$ and assignment $b \in \mathcal{A}^2$ such that $\{i, j\}$ strongly blocks a at \succsim via b . Note that if patients' preferences over desirable kidneys are strict, the definitions of weak and strong blocking are equivalent and the strict and the weak core coincide.

4.3 The Natural Age Structure Π^* : Strict Preferences

In this section, we analyze the natural age structure Π^* that induces strict preferences over desirable kidneys. This case provides interesting insights for the general case with arbitrary age structures.

According to Π^* , each age group contains a single kidney and for each patient i , $\Pi^*(i) = \{\omega_i\}$. In this case, only the natural ordering σ^* respects Π^* . We denote with ψ^* the **age based priority rule according to the natural order** σ^* . That is, $\psi^* \equiv \psi^{\sigma^*}$. Note that for each $\succsim \in \mathcal{D}^{\Pi^*}$, $\mathcal{M}_n^{\sigma^*}(\succsim)$ is single-valued. Thus, ψ^* is the unique age based priority rule. In the following result, we state the close relation between ψ^* and the strict core in the domain \mathcal{D}^{Π^*} .

Proposition 1. *For each $\succsim \in \mathcal{D}^{\Pi^*}$, $\mathcal{C}(\succsim) = \{\psi^*(\succsim)\}$.*

With Proposition 1 at hand, we characterize the age based priority rule ψ^* as the unique pairwise exchange rule that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} .

Theorem 2. *A rule $\varphi : \mathcal{D}^{\Pi^*} \rightarrow A^2$ satisfies individual rationality, 2-efficiency and strategy-proofness if and only if φ is the age based priority rule ψ^* .*

The proof of Theorem 2 exploits a general result for exchange economies with indivisible goods obtained by Sönmez [36] on the relation between the existence of single-valued strict cores and the existence of rules that satisfy *strategy-proofness*. It is not difficult to prove that ψ^* satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} . By Proposition 1, ψ^* always selects the unique assignment in the strict core of the associated pairwise exchange problem. Then, by Theorem 1 in [36], ψ^* is the only rule that satisfies the proposed properties.

4.4 General Age Structures

In this section we consider arbitrary age structures $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ with $m \geq 2$. Under an arbitrary age structure, age based priority rules retain many of the properties that they possess in \mathcal{D}^{Π^*} . By their very definition, age based priority rules satisfy *individual rationality* and *2-efficiency*. Although the strict core of the associated pairwise exchange problem may be empty for some preference profile, every age based priority rule selects assignments in the weak core.¹⁷ In the following theorem, we show that age based priority rules satisfy *strategy-proofness*, and that they are the only priority rules that fulfill this property.¹⁸

Theorem 3. *For each age structure Π and each priority ordering σ , the priority rule ψ^σ satisfies strategy-proofness on \mathcal{D}^Π if and only if ψ^σ is an age based priority rule.*

We devote the remainder of this section to providing further evidence of the central position of age based priority rules among the rules that satisfy *individual rationality*, *2-efficiency*, and *strategy-proofness* for arbitrary age based domains. We focus on rules

¹⁷For each age structure Π , the associated Π age based domain satisfies the non-odd-cycle condition [7]. Thus, the weak core of the associated pairwise exchange problem is never empty.

¹⁸In fact, age based priority rules also satisfy the stronger property of *weak coalitional strategy-proofness* in \mathcal{D}^Π . *Weak coalitional strategy-proofness* requires that for each $\succsim \in \mathcal{D}$, there does not exist $T \subseteq N$ and $\succsim' \in \mathcal{D}^\Pi$ such that for each $i \notin T$ $\succsim'_i = \succsim_i$, and for each $j \in T$ $\varphi_j(\succsim') \succ_j \varphi_j(\succsim)$. In our framework, stronger definitions of coalitional *strategy-proofness* are incompatible with the remaining properties [10].

that satisfy the additional property of *non-bossiness*. Clearly, age based priority rules satisfy *non-bossiness*. In the following lemma, we state an important feature of the rules that satisfy *non-bossiness* together with our standard properties.

Lemma 1. *If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for each $i, j \in N$ and $\succsim \in \mathcal{D}^\Pi$, $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ imply either $\varphi_i(\succsim) \succsim_i \omega_j$ or $\varphi_j(\succsim) \succsim_j \omega_i$ (or both).*

An immediate consequence of Lemma 1 is that our properties preclude the maximization of the number of patients that receive a desirable kidney. This is in sharp contrast with the results in the dichotomous domain framework presented by Roth et al. [31]. Every pairwise exchange rule defined in the domain \mathcal{D}^Π that satisfies *individual rationality* and *2-efficiency* maximizes the number of patients who receive a desirable transplant. We illustrate this fact in the following example.

Example 1. *Let $N = \{1, 2, 3, 4\}$, $\Pi(1) = \{\omega_1, \omega_2\}$, and $\Pi(2) = \{\omega_3, \omega_4\}$. Consider the preference profile $\succsim \in \mathcal{D}^\Pi$ such that $D(\succsim_1) = \{\omega_2, \omega_3, \omega_4\}$, $D(\succsim_2) = \{\omega_1, \omega_3, \omega_4\}$, $D(\succsim_3) = \{\omega_1\}$, and $D(\succsim_4) = \{\omega_2\}$. Let $a = (\omega_3, \omega_4, \omega_1, \omega_2)$. According to a , four transplantations are performed and $a \in \mathcal{I}(\succsim)$. By Lemma 1, however, for each rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ that satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, $\varphi(\succsim) = (\omega_2, \omega_1, \omega_3, \omega_4)$.*

Adding *non-bossiness* to the list of desirable properties for rules has a second important consequence. A rule that satisfies our list of properties always selects assignments in the weak core of the associated pairwise exchange problem.

Proposition 2. *If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for each $\succsim \in \mathcal{D}^\Pi$, $\varphi(\succsim) \in \tilde{\mathcal{C}}(\succsim)$.*

While a full characterization of the rules that satisfy *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*, seems to be out of reach, we can provide further insights into the structure of such rules. Lemma 1 and Example 1 suggest, that according to a rule that satisfies the above properties, the number of transplants involving pairs with donors in $\Pi(1)$ is maximized. We define a new class of pairwise exchange rules that extend this intuition to all age groups.

For each $a \in \mathcal{A}^2$, each $t, t' \in \mathbb{N}$ with $t \leq t'$ and $t' \leq m$, let

$$M_{t,t'}(a) \equiv \left\{ i \in N \left| \begin{array}{l} a_i \neq \omega_i \text{ \& } \text{either } \omega_i \in \Pi(t) \text{ and } a_i \in \Pi(t') \\ \text{or } \omega_i \in \Pi(t') \text{ and } a_i \in \Pi(t) \end{array} \right. \right\}.$$

That is, $M_{t,t'}(a)$ contains patients with a donor in $\Pi(t)$ who receive a kidney in $\Pi(t')$ and patients with a donor in $\Pi(t')$ who receive a kidney in $\Pi(t)$. For each $a \in \mathcal{A}^2$, let $P_{1,1}(a) \equiv \emptyset$ and define recursively for each $t, t' \in N$, such that $\{t, t'\} \neq \{1, 1\}$, $t \leq t'$ and $t' \leq m$:

$$P_{t,t'}(a) \equiv \begin{cases} P_{t,t'-1}(a) \cup M_{t,t'-1}(a) & \text{if } t < t', \\ P_{t-1,m}(a) \cup M_{t-1,m}(a) & \text{if } t = t'. \end{cases}$$

Finally, for each $a \in \mathcal{A}^2$ and each $t, t' \in \mathbb{N}$ with $t \leq t' \leq m$, let

$$Q_{t,t'}(a) = \{a' \in \mathcal{A}^2 \mid \text{for each } i \in P_{t,t'}(a), a'_i = a_i\}.$$

A rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ is a ***sequentially maximizing exchange rule*** if for each $\succsim \in \mathcal{D}^\Pi$,

- (i) $\varphi(\succsim) \in \mathcal{I}(\succsim)$, and
- (ii) for each $t, t' \in \mathbb{N}$ with $t \leq t' \leq m$, for each $a \in Q_{t,t'}(\varphi(\succsim)) \cap \mathcal{I}(\succsim)$,

$$\#M_{t,t'}(\varphi(\succsim)) \geq \#M_{t,t'}(a).$$

Sequentially maximizing exchange rules propose assignments that maximize the number of exchanges within each age group and among different age groups following the priority orderings induced by the age structure. These rules generalize one of the most important features of the age based priority rules. According to sequentially maximizing exchange rules, transplantations involving the most valuable kidneys are proposed first, and then an iterative process follows with regard to the remaining age groups. To illustrate the logic behind sequentially maximizing exchange rules, consider an initial age structure with only two age groups $\Pi = \{\Pi(1), \Pi(2)\}$. Thus, $\Pi(1)$ is the set of young donors, and $\Pi(2)$ is the set of mature donors. In this case, a sequential matching maximizing rule maximizes among the assignments in $\mathcal{I}(\succsim)$ the number of transplants involving only donor-patient pairs with young donors. Then, given the exchanges between pairs

of patients with a young donor, it maximizes the number of swaps between pairs with a young donor and pairs with a mature donor. Finally, given the exchanges arranged in the previous stages, the rule maximizes the number of swaps involving donor-patient pairs with mature donors. For more general age structures, a sequential matching maximizing rule proceeds sequentially in the same fashion. In the first stage, it maximizes the number of exchanges among pairs with the youngest donors. In a second stage, given the exchanges selected in the first stage, it maximizes the number of exchanges among pairs with the youngest donors and pairs with donors in the second age group. If there are more than two age groups, a sequential maximizing rule continues in the same fashion, maximizing the number of exchanges among pairs with the youngest donors and pairs with donors in the third age group, and in subsequent stages with the remaining age groups. Then it applies the same logic with pairs in the second age group and so on.

In the following theorem, we provide our more general result.

Theorem 4. *If $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then φ is a sequential matching maximizing rule.*

Before concluding, it is worth noting that Theorem 4 implies that there may exist assignments in the weak core that are never selected by a rule that satisfies our list of properties.¹⁹

4.5 Extensions

In this section, we briefly discuss the robustness of our results and possible extensions of our analysis.

In this paper, we have focused on a very narrow domain of preferences justified by the PKE application. It is natural to consider the possibility of extending the results to less

¹⁹Assume $N = \{1, 2, 3, 4, 5, 6\}$ and $\Pi = \{\Pi(1), \Pi(2)\} = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_1) = \{\omega_2, \omega_3\}$, $D(\succsim_2) = \{\omega_1, \omega_4\}$, $D(\succsim_3) = \{\omega_1, \omega_5\}$, $D(\succsim_4) = \{\omega_2, \omega_6\}$, $D(\succsim_5) = \{\omega_3\}$, and $D(\succsim_6) = \{\omega_4\}$. Let $a \in \mathcal{A}^2$ be such that $a = (\omega_2, \omega_1, \omega_5, \omega_6, \omega_3, \omega_4)$. It is easy to check that $a \in \tilde{\mathcal{C}}(\succsim)$. According to a , there are three compatible donor swaps, but only one between patients with a donor in $\Pi(1)$. A rule that satisfies our properties cannot choose a at \succsim because it is possible to carry out two exchanges involving donor-patients pairs in $\Pi(1)$. (For instance, consider $b = (\omega_3, \omega_4, \omega_1, \omega_2, \omega_5, \omega_6)$.)

restricted domains of preferences. We can prove that for all preference domains that admit cyclical profiles in the sense that there are $i, j, k \in N$ such that $\omega_j \succ_i \omega_k \succ_j \omega_i \succ_k \omega_j$, there is no pairwise exchange rule that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* (Theorem 1, [21]). Unfortunately, the most natural extensions of age based preferences generate such cyclical profiles. For instance, suppose that the quality of kidneys depends on two observable characteristics that are not perfectly correlated: age and health status. Kidneys can be ordered according to each of those characteristics, but the relative importance of one characteristic with respect to the other is private information. That is, each patient prefers a (compatible) kidney from a young and healthy donor to an organ from a mature or unhealthy donor. However, two patients may differently rank a kidney from a young donor in poor health condition and a kidney from a mature donor in perfect health. If patients do not coincide on the characteristic most relevant to defining their preferences, then the implicit preference domain admits cyclical profiles.²⁰ What constitute the precise requirements on the domain of preferences that make our properties compatible is an interesting open question.

We have assumed throughout the paper that each patient's set of desirable kidneys is private information of each patient and only the age structure is public information. However, since compatibility issues are determined by blood tests directly carried out by the PKE programs, transplant coordinators have access to all the relevant medical information on donors and patients [2, 3, 21]. Thus, it is natural to consider a model where patients (or their doctors) do not report their sets of compatible kidneys because it is public information. In this scenario, there is a crucial item of information that still remains private: the minimum quality of the compatible kidney required by a patient to undergo transplantation. The introduction of additional information on the compatibility issue would correspond to a framework where preferences are further restricted. The analysis of this setting entails cumbersome notation and the results depend on the compatibility information, but the intuition of our result and our main findings are robust to the introduction of public information about donor–patient compatibility.

Finally, a natural extension of our model considers the possibility that patients may

²⁰If all patients consider the same characteristic as the most important one, then there would be an age structure Π such that patients' preferences are Π age based.

find more than one potential donor. Even if only one among the potential donors of a patient donates her kidney, the fact that a patient may have many potential donors can greatly increase the chances of finding mutually compatible pairs. The analysis and conclusions of this model are similar to the one donor case. The fact that a patient may find potential donors in different age groups highlights that the priority in the exchange process is not assigned to patients with younger donors. Instead, the priority algorithm looks for matches only involving younger donors first, and then sequentially follows the same process among the donors of the different age groups. Notably, the extension of age based priority rules to the multiple donors scenario does not provide incentives for the patients to manipulate by withholding some potential donors.

5 Conclusion

We have analyzed a framework that incorporates two important features of PKE, namely, the existence of feasibility constraints in the proposed assignments and the existence of observable characteristic of kidneys – donors’ age – that affect the expected living standards of the recipients and consequently patients’ preferences over outcomes. In this model, we have shown that only PKE rules restricted to select pairwise assignments, and which assign kidneys according to a priority based on donors’ ages, satisfy *individual rationality*, *efficiency*, and *strategy-proofness*. We now briefly discuss now some implications of our model for the design of PKE programs.

The main justification for our framework is the design of PKE protocols that encourage the participation of compatible donor-patient pairs. In the standard model, the motivation of compatible donor-patient pairs to participate in PKE programs is entirely altruistic ([32, 37]). In our model, compatible donor-patient pairs with relatively old donors may be willing to enroll PKE programs that adopt age based priority rules because their patients can receive organs with longer expected graft survival. In fact, the participation of compatible pairs may represent the most important factor in expanding the number of kidney paired exchanges. A recent study by Gentry et al. [11] uses simulated data to prove that there may be large benefits for both incompatible pairs and compatible pairs if compatible pairs are willing to participate in PKE programs. The participation

of compatible pairs could dramatically reduce blood group imbalances in the pool of compatible pairs.²¹ As a result, the match rate for incompatible pairs would double from 28.2 to 64.5 percent for a single-center program and from 37.4 to 75.4 percent for a national program. Therefore, the positive impact of allowing compatible pairs to participate in PKE programs could offset the efficiency loss due to the restriction to pairwise assignments.

6 Appendix: Proofs

Proof of Theorem 1. The arguments follow the proof of Theorem 1 in Nicolò and Rodríguez-Álvarez [21]. We include the complete proof for the sake of completeness.

Let $3 \leq k \leq n - 1$. Assume, by way of contradiction, that there are a partition Π and a rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^k$ such that φ satisfies *individual rationality*, *k-efficiency*, and *strategy-proofness* in \mathcal{D}^Π . Without loss of generality, by Assumption A, let $\omega_1 \in \Pi(l)$, $\omega_2 \in \Pi(l')$, $\omega_k \in \Pi(\bar{l})$ and $\omega_{k+1} \in \Pi(\bar{l}')$ with $l < l'$ and $\bar{l} < \bar{l}'$.²² Let $\succsim \in \mathcal{D}^\Pi$ be such that for each $i \notin \{k - 1, k + 1\}$, $D(\succsim_i) = \{\omega_{i+1}\}$, $D(\succsim_{k-1}) = \{\omega_k, \omega_{k+1}\}$, and $D(\succsim_{k+1}) = \{\omega_1, \omega_2\}$.

Let $\succsim' \in \mathcal{D}^\Pi$ be such that for each $i \neq k - 1$, $\succsim_i = \succsim'_i$, and $D(\succsim'_{k-1}) = \{\omega_k\}$. Under profile \succsim' , by *individual rationality*, either no object is assigned to any patient $1, \dots, k + 1$, or patient $k + 1$ receives ω_2 , patient 1 receives ω_1 , and every other patient i receives ω_{i+1} (the kidney of her next to the right neighbor). By *k-efficiency*:

$$\varphi(\succsim') = \begin{bmatrix} (1, \omega_1), \\ (i, \omega_{i+1}), \quad \forall i = 2, \dots, k \\ (k + 1, \omega_2) \end{bmatrix}.$$

²¹Most of group-0 donors can directly donate to their intended recipients. Hence, group-0 patients in the incompatible donor-patient pool must rely on a scant number of group-0 donors and can rarely find a match.

²²For instance, we can assume that $\omega_1 \in \Pi(1)$ and $\{\omega_2, \omega_k\} \in \Pi(2)$, and $\omega_{k+1} \in \Pi(3)$. Alternatively, we can have $\{\omega_1, \omega_k\} \subseteq \Pi(1)$ and $\{\omega_2, \omega_{k+1}\} \subseteq \Pi(2)$, and apply a convenient relabeling of patients and donors in order to satisfy our notational assumption.

By *strategy-proofness*, $\varphi_{k-1}(\succsim) \succsim_{k-1} \varphi_{k-1}(\succsim') = \omega_k$. Note that, according to \succsim_{k-1} , ω_k is patient $k-1$'s preferred kidney. Then, $\varphi_{k-1}(\succsim) = \omega_k$. By *k-efficiency* and *individual rationality*, $\varphi(\succsim) = \varphi(\succsim')$.

Let $\succsim'' \in \mathcal{D}^\Pi$ be such that for each $i \neq k+1$, $\succsim_j = \succsim_j''$ and $D(\succsim_{k+1}'') = \{\omega_1\}$. The same arguments we employed to determine $\varphi(\succsim')$ apply here to obtain:

$$\varphi(\succsim'') = \begin{bmatrix} (i, \omega_{i+1}) \text{ (modulo } k+1), & \forall i \notin \{k, k-1\} \\ (k-1, \omega_{k+1}), \\ (k, \omega_k) \end{bmatrix}.$$

Note that $\omega_1 = \varphi_{k+1}(\succsim'') = \varphi_{k+1}(\succsim_{k+1}'', \succsim_{-(k+1)}) \succ_{k+1} \varphi_{k+1}(\succsim) = \omega_2$, which contradicts *strategy-proofness*. \square

Proof of Proposition 1. Let $\succsim \in \mathcal{D}^{\Pi*}$. We first prove that for each $b \in \mathcal{A}^2 \setminus \psi^*(\succsim)$, $b \notin \mathcal{C}(\succsim)$. If $b \notin \mathcal{I}(\succsim)$, then clearly $b \notin \mathcal{C}(\succsim)$. Let $b \in \mathcal{I}(\succsim)$. Let i be the patient such that $b_i \neq \psi_i^*(\succsim)$ and for each $i' \in N$ with $i' < i$, $b_{i'} = \psi_{i'}^*(\succsim)$. Let j be the patient such that $\psi_j^*(\succsim) = \omega_j$. Note that since $b \in \mathcal{I}(\succsim)$ and preferences over desirable kidneys are strict, by the definitions of ψ^* and i , we have $i < j$ and $\psi_i^*(\succsim) = \omega_j \succ_i b_i$. Because $\psi_j^*(\succsim) = \omega_i \in D(\succsim_j)$, for each $i' \in N$ such that $\omega_{i'} \succ_j \omega_i$, $i' < i$. Thus, since for each $i' < i$, $\psi_{i'}^*(\succsim) = b_{i'} \neq \omega_j$, we have $\psi_j^*(\succsim) = \omega_i \succ_j b_j$. Hence, the pair $\{i, j\}$ weakly blocks b at \succsim via $\psi^*(\succsim)$.

To conclude the proof, we show that $\psi^*(\succsim) \in \mathcal{C}(\succsim)$. Since $\psi^*(\succsim) \in \mathcal{I}(\succsim)$, there is no single-patient that weakly blocks $\psi^*(\succsim)$ at \succsim . Hence, we check that $\psi^*(\succsim)$ is not weakly blocked by a pair of patients. Assume to the contrary that there are a pair $\{i, j\} \subset N$ and $a \in \mathcal{A}^2$ such that $\{i, j\}$ weakly blocks $\psi^*(\succsim)$ at \succsim via a . Without loss of generality, let $i < j$. Since preferences over desirable kidneys are strict and $a \neq \psi^*(\succsim)$, $a_i = \omega_j \succ_i \psi_i^*(\succsim)$ and $a_j = \omega_i \succ_j \psi_j^*(\succsim)$. Let j' be the patient such that $\psi_{j'}^*(\succsim) = \omega_{j'}$. There are two cases:

Case i). $j' < i$. Because $\succsim_i \in \mathcal{D}_i^{\Pi*}$ and $j' < i < j$, $\omega_{j'} \succ_i \omega_j = a_i$, which contradicts $\{i, j\}$ strongly blocking $\psi^*(\succsim)$ via a .

Case ii). $i \leq j'$. By the definition of ψ^* , either $\omega_i \notin D(\succsim_j)$ or there is i' with $i' < i$, such that $\psi_j^*(\succsim) = \omega_{i'}$. In either case, $\psi_j^*(\succsim) \succ_j \omega_i$, which contradicts $\{i, j\}$ weakly blocking $\psi^*(\succsim)$ via a .

Since both cases are exhaustive, $\psi^*(\succsim) \in \mathcal{C}(\succsim)$, and by the arguments in the previous paragraph, $\mathcal{C}(\succsim) = \{\psi^*(\succsim)\}$. \square

Proof of Theorem 2. It is easy to check that ψ^* satisfies *individual rationality* and *2-efficiency*. Consequently, we only show that ψ^* satisfies *strategy-proofness*. Let $i \in N$, $\succsim \in \mathcal{D}^{\Pi^*}$ and $\succsim'_i \in \mathcal{D}_i^{\Pi^*}$. Assume first that, $\psi_i^*(\succsim) = \psi_i^*(\succsim'_i, \succsim_{-i})$, then $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$. Assume now that $\psi_i^*(\succsim) \neq \psi_i^*(\succsim'_i, \succsim_{-i})$. Let $j, j' \in N$ be such that $\psi_i^*(\succsim) = \omega_j$ and $\psi_i^*(\succsim'_i, \succsim_{-i}) = \omega_{j'}$. There are two cases:

Case i). $i = j$. By the definition of ψ^* , $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, and $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$.

Case ii). $i \neq j$. By the definition of $\psi^*(\succsim)$, $\omega_j \in D(\succsim_i)$ and therefore $\omega_j \succ_i \omega_i$. Since $\psi_i^*(\succsim) \neq \psi_i^*(\succsim'_i, \succsim_{-i})$, then either $j' < j$ with $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, or $j < j'$, or $i = j'$. In either case $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$.

Since both cases are exhaustive, ψ^* satisfies *strategy-proofness*.

Next, we prove the necessity side. Let φ be a rule that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} . Note that for each $i \in N$ and $\succsim_i \in \mathcal{D}_i^{\Pi^*}$, by the definition of age based preferences, for each $a \in \mathcal{A}^2$, $a_i \sim_i \omega_i$ if and only if $a_i = \omega_i$. Moreover, for each $a \in \mathcal{A}^2$ such that $a_i \succsim_i \omega_i$, there is $\succsim'_i \in \mathcal{D}_i^{\Pi^*}$ such that $D(\succsim'_i) = \{b_i \in D(\succsim_i) \mid b_i \succsim_i a_i\}$. Hence, the domain \mathcal{D}^{Π^*} satisfies Assumptions A and B on the domain of preferences proposed by Sönmez [36]. By Proposition 1, for each $\succsim \in \mathcal{D}^{\Pi^*}$, $\mathcal{C}(\succsim) = \{\psi^*(\succsim)\} \neq \emptyset$. By [36, Theorem 1], if there is a rule φ that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} , then for each $\succsim \in \mathcal{D}^{\Pi^*}$, $\varphi(\succsim) \in \mathcal{C}(\succsim)$. By the arguments in the previous paragraph, ψ^* satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in \mathcal{D}^{Π^*} . Thus, $\varphi = \psi^*$. \square

The following remark and lemma are useful in the proof of Theorem 3. If the last element of the age structure contains a single kidney ($\Pi(m) = \{\omega_n\}$), then there exist priority orderings that do not respect the partition but do define priority rules that are

equivalent to an age based priority rule. This situation may arise when the priority orders of patient n and one patient with her donor's kidney in $\Pi(m-1)$ are switched with respect to the priorities assigned by a priority order that respects Π .

Remark 1. For each age structure $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ and each priority order σ that does not respect Π , then

- (i) either there are $i, j, h \in N$ and $l, l' \leq m$ such that $\omega_i \in \Pi(l)$, $\omega_j \in \Pi(l')$ $l < l'$, and $\sigma(j) < \sigma(i)$ and $\sigma(j) < \sigma(h)$,
- (ii) or there are $i, j \in N$ such that $\omega_i \in \Pi(m-1)$, $\{\omega_j\} = \Pi(m)$, $\sigma(j) = (n-1)$ and $\sigma(i) = n$.

Lemma 2. Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be such that $\#\Pi(m) = 1$. Let the priority ordering σ respect Π and let $i, j \in N$ be such that $\omega_i \in \Pi(m-1)$, $\{\omega_j\} = \Pi(m)$, and $\sigma(i) = (n-1)$. If the priority ordering σ' is such that for each $h \notin \{i, j\}$, $\sigma'(h) = \sigma(h)$, $\sigma'(j) = (n-1)$, and $\sigma'(i) = n$, then for each $\succsim \in \mathcal{D}^\Pi$, $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_n^{\sigma'}(\succsim)$.

Proof. Let $\succsim \in \mathcal{D}^\Pi$. Note that for each $t \leq (n-2)$, $\mathcal{M}_t^\sigma(\succsim) = \mathcal{M}_t^{\sigma'}(\succsim)$. For each $h \notin \{i, j\}$ and each $a, a' \in \mathcal{M}_{n-2}^\sigma(\succsim)$, $a_h \sim_h a'_h$. There are three cases:

Case i). There are $h \notin \{i, j\}$ and $a \in \mathcal{M}_{n-2}^\sigma(\succsim)$ such that $a_h = \omega_j$. Since $\Pi(m) = \{\omega_j\}$, for each $b, b' \in \mathcal{M}_{n-2}^\sigma(\succsim)$, $b_h = b'_h = \omega_j$, and $b_j = b'_j = \omega_h$ and $b_j \sim_j b'_j$. Note that $\mathcal{M}_{n-1}^\sigma(\succsim) = \{a \in \mathcal{M}_{n-2}^\sigma(\succsim) \mid \text{for no } b \in \mathcal{M}_{n-2}^\sigma(\succsim), b_i \succ_i a_i\}$. Since $\mathcal{M}_{n-1}^\sigma(\succsim) \subseteq \mathcal{M}_{n-2}^\sigma(\succsim)$ and for each $b, b' \in \mathcal{M}_{n-2}^\sigma(\succsim)$, $b_j \sim_j b'_j$, $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_{n-1}^\sigma(\succsim)$. With the same arguments, $\mathcal{M}_{n-1}^{\sigma'}(\succsim) = \mathcal{M}_{n-2}^{\sigma'}(\succsim)$, and $\mathcal{M}_n^{\sigma'}(\succsim) = \{a \in \mathcal{M}_{n-2}^{\sigma'}(\succsim) \mid \text{for no } b \in \mathcal{M}_{n-2}^{\sigma'}(\succsim), b_i \succ_i a_i\}$. Since $\mathcal{M}_{n-2}^\sigma(\succsim) = \mathcal{M}_{n-2}^{\sigma'}(\succsim)$, we have $\mathcal{M}_n^\sigma(\succsim) = \mathcal{M}_n^{\sigma'}(\succsim)$.

Case ii). There is $a \in \mathcal{M}_{n-2}^\sigma(\succsim)$ such that $a_j = \omega_i$. By Case i), for each $b \in \mathcal{M}_{n-2}^\sigma(\succsim)$, $b_j \in \{\omega_i, \omega_j\}$. First, we prove that for each $b \in \mathcal{M}_{n-2}^\sigma(\succsim)$, $b_i \in \{\omega_i, \omega_j\}$. Let $a \in \mathcal{M}_{n-2}^\sigma(\succsim)$ with $a_j = \omega_i$. Assume to the contrary that there is $h_1 \notin \{i, j\}$ such that $b_{h_1} = \omega_i$. Let $\omega_{h_1} \in \Pi(l)$ ($l \leq m-1$.) Assume that $a_{h_1} \notin \Pi(m-1) \setminus \{\omega_i\}$, then $a_{h_1} \succ_{h_1} b_{h_1}$, which is a contradiction since $a, b \in \mathcal{M}_{n-2}^\sigma(\succsim)$ and $h_1 \notin \{i, j\}$. Hence, $a_{h_1} \in \Pi(m-1) \setminus \{\omega_i\}$. Let $i_1 \notin \{i, j, h_1\}$ be such that $\omega_{i_1} = a_{h_1}$. Assume

that $b_{i_1} \notin \Pi(l) \setminus \{\omega_{k_1}\}$, then either $a_{i_1} \succ_{i_1} b_{i_1}$ or $b_{i_1} \succ_{i_1} a_{i_1}$, which is a contradiction since $a, b \in \mathcal{M}_{n-2}^\sigma(\succ)$ and $i_1 \notin \{i, j\}$. Therefore, $b_{i_1} \in \Pi(l) \setminus \{\omega_{h-1}\}$. Since $\Pi(l)$ and $\Pi(m-1)$ are finite sets, we can continue in the same fashion until we eventually reach a contradiction. That is, there is $t \geq 2$ such that either $a_{h_t} \notin \Pi(m-1) \setminus \{\omega_i, \omega_{i_1}, \dots, \omega_{i_{t-1}}\}$ or $b_{i_t} \notin \Pi(l) \setminus \{\omega_{h_1}, \dots, \omega_{h_t}\}$. Hence, for each $a \in \mathcal{M}_{n-2}^\sigma(\succ)$, $a_i \in \{\omega_i, \omega_j\}$.

To conclude, since $\mathcal{M}_{n-2}^\sigma(\succ) \subseteq \mathcal{I}(\succ)$, if there is $a \in \mathcal{M}_{n-2}^\sigma(\succ)$ such that $a_j = \omega_i$, then $\omega_i \in D(\succ_j)$ and $\omega_j \in D(\succ_i)$. Because for each $b \in \mathcal{M}_{n-2}^\sigma(\succ)$, $b_i \in \{\omega_i, \omega_j\}$ and $b_j \in \{\omega_i, \omega_j\}$, we have $\mathcal{M}_{n-1}^\sigma(\succ) = \{a \in \mathcal{M}_{n-2}^\sigma(\succ) \mid a_i = \omega_j\} = \mathcal{M}_n^\sigma(\succ)$. Analogously, $\mathcal{M}_{n-1}^{\sigma'}(\succ) = \{a \in \mathcal{M}_{n-2}^\sigma(\succ) \mid a_j = \omega_i\} = \mathcal{M}_n^{\sigma'}(\succ)$. Thus, $\mathcal{M}_n^\sigma(\succ) = \mathcal{M}_n^{\sigma'}(\succ)$.

Case iii). For each $a \in \mathcal{M}_{n-2}^\sigma(\succ)$, we have that $a_j = \omega_j$. Then, for each $a, b \in \mathcal{M}_{n-2}^\sigma(\succ)$, $a_j \sim_j b_j$. Applying the arguments in Case i), we obtain $\mathcal{M}_n^\sigma(\succ) = \mathcal{M}_n^{\sigma'}(\succ)$.

□

Proof of Theorem 3. Consider a partition Π , we prove that if σ respects Π , then ψ^σ satisfies *strategy-proofness*. The proof replicates with minimal variations the arguments in the proof of Theorem 2 but we include them for the sake of completeness. To simplify notation, we consider the natural priority ordering σ^* . The arguments apply directly for every arbitrary priority ordering that respects Π . Let $i \in N$, $\succ \in \mathcal{D}^\Pi$ and $\succ'_i \in \mathcal{D}_i^\Pi$. Assume first that, $\psi_i^*(\succ) = \psi_i^*(\succ'_i, \succ_{-i})$, then $\psi_i^*(\succ) \sim_i \psi_i^*(\succ'_i, \succ_{-i})$. Assume now that that $\psi_i^*(\succ) \neq \psi_i^*(\succ'_i, \succ_{-i})$. Let $j, j' \in N$ be such that $\psi_i^*(\succ) = \omega_j$ and $\psi_i^*(\succ'_i, \succ_{-i}) = \omega_{j'}$. There are two cases:

Case i). $i = j$. By the definition of ψ^* , $\omega_{j'} \in D(\succ'_i) \setminus D(\succ_i)$, and $\psi_i^*(\succ) \sim_i \psi_i^*(\succ'_i, \succ_{-i})$.

Case ii). $i \neq j$. Assume first that $i = j'$. Since $\psi_i^*(\succ) = \omega_j \neq \omega_i = \psi_i^*(\succ'_i, \succ_{-i})$, $\omega_j \in D(\succ_i)$, and therefore $\psi_i^*(\succ) \sim_i \psi_i^*(\succ'_i, \succ_{-i})$. Finally, assume that $i \neq j'$. In this case, there are $l, l' \in \mathbb{N}$ such that $\psi_i^*(\succ) = \omega_j \in \Pi(l)$ and $\psi_i^*(\succ'_i, \succ_{-i}) = \omega_{j'} \in \Pi(l')$. By the definition of $\psi^*(\succ)$, either $l' < l$ with $\omega_{j'} \in D(\succ'_i) \setminus D(\succ_i)$, or $l \leq l'$. In either case $\psi_i^*(\succ) \sim_i \psi_i^*(\succ'_i, \succ_{-i})$.

Since both cases are exhaustive, ψ^* satisfies *strategy-proofness*.

Next, consider a priority ordering σ such that the priority rule ψ^σ is not an age based priority rule. Hence, the priority ordering σ does not respect Π . By Remark 1 and Lemma 2, there are $i, j, h \in N$, and $l, l' \leq m$ such that $\omega_i \in \Pi(l)$, $\omega_j \in \Pi(l')$, $l < l'$, $\sigma(j) < \sigma(i)$ and $\sigma(j) < \sigma(h)$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_i) = \{\omega_h\}$, $D(\succsim_j) = \{\omega_h\}$, and $D(\succsim_h) = \{\omega_i, \omega_j\}$. Clearly, $\psi_h^\sigma(\succsim) = \omega_j$. Let $\succsim'_h \in \mathcal{D}_h^\Pi$ be such that $D(\succsim'_h) = \{\omega_i\}$. Then, by 2-efficiency, $\psi_h^\sigma(\succsim'_h, \succsim_{-h}) = \omega_i$, and $\psi_m^\sigma(\succsim'_h, \succsim_{-h}) \succ_h \psi_h^\sigma(\succsim)$, which proves that ψ^σ violates *strategy-proofness*. \square

Proof of Lemma 1. Assume to the contrary that there are $i, j \in N$ and $\succsim \in \mathcal{D}^\Pi$ such that there exist $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ but $\omega_i \succ_j \varphi_j(\succsim)$ and $\omega_j \succ_i \varphi_i(\succsim)$. Let $\succsim^1 \in \mathcal{D}^\Pi$ be such that $\succsim_i^1 = \succsim_i$, $\succsim_j^1 = \succsim_j$, and for each $h \in N \setminus \{i, j\}$, $D(\succsim_h^1) = \{\varphi_h(\succsim)\}$. Let $h \in N \setminus \{i, j\}$. By *individual rationality*, $\varphi_h(\succsim_h^1, \succsim_{-h}) \in \{\omega_h, \varphi_h(\succsim)\}$. By *strategy-proofness*, $\varphi_h(\succsim_h^1, \succsim_{-h}) \succsim_h^1 \varphi_h(\succsim)$. Hence, $\varphi_h(\succsim_h^1, \succsim_{-h}) = \varphi_h(\succsim)$, and by *non-bossiness*, $\varphi(\succsim_h^1, \succsim_{-h}) = \varphi(\succsim)$. Repeating the argument with the remaining patients (one at a time), we obtain $\varphi(\succsim^1) = \varphi(\succsim)$. Let $\succsim^2 \in \mathcal{D}^\Pi$ be such that $D(\succsim_i^2) = \{\omega_j, \varphi_i(\succsim)\}$, $D(\succsim_j^2) = \{\omega_i, \varphi_j(\succsim)\}$ and for each patient $h \notin \{i, j\}$, $\succsim_h^2 = \succsim_h^1$. By *individual rationality* and *strategy-proofness*, $\varphi_i(\succsim_i^2, \succsim_{-i}^1) = \varphi_i(\succsim^1)$. By *non-bossiness*, $\varphi(\succsim_i^2, \succsim_{-i}^1) = \varphi(\succsim^1)$. Similarly, by *individual rationality* and *strategy-proofness*, $\varphi_j(\succsim_j^2, \succsim_{-j}^1) = \varphi_j(\succsim^1)$. By *non-bossiness*, $\varphi(\succsim_j^2, \succsim_{-j}^1) = \varphi(\succsim^1)$. Thus, $\varphi(\succsim^2) = \varphi(\succsim^1)$. Let $\succsim^3 \in \mathcal{D}^\Pi$ be such that $D(\succsim_i^3) = \{\omega_j\}$, $D(\succsim_j^3) = \{\omega_i\}$, and for each patient $h \notin \{i, j\}$, $\succsim_h^3 = \succsim_h^2$. By *individual rationality* and *strategy-proofness*, $\varphi_i(\succsim_i^3, \succsim_{-i}^2) = \{\omega_i\}$. By *individual rationality*, $\varphi_j(\succsim_j^3, \succsim_{-j}^2) \in \{\omega_j, \varphi_j(\succsim^2)\}$. By 2-efficiency, $\varphi_j(\succsim_j^3, \succsim_{-j}^2) = \varphi_j(\succsim^2)$. Finally, by *individual rationality* and *strategy-proofness*, $\varphi_j(\succsim_{\{i,j\}}^3, \succsim_{-\{i,j\}}^2) = \{\omega_j\}$ and $\varphi_i(\succsim_{\{i,j\}}^3, \succsim_{-\{i,j\}}^2) = \{\omega_i\}$, which violates 2-efficiency. \square

Proof of Proposition 2. Let $\succsim \in \mathcal{D}^\Pi$. Assume to the contrary that φ satisfies *individual rationality*, 2-efficiency, *strategy-proofness*, and *non-bossiness*, but $\varphi(\succsim) \notin \tilde{\mathcal{C}}(\succsim)$. Since φ satisfies *individual rationality*, and $\varphi(\succsim) \in \mathcal{I}(\succsim)$, there are an assignment $a \in \mathcal{A}^2$, and a pair $\{i, j\} \subset N$ such that the pair $\{i, j\}$ strongly blocks $\varphi(\succsim)$ at \succsim via a . Since φ satisfies

individual rationality, $a_i = \omega_j \in D(\succsim_i)$, and $a_j = \omega_i \in D(\succsim_j)$. Because, $\{i, j\}$ strongly blocks $\varphi(\succsim)$ at \succsim via a , $\omega_j \succ_i \varphi_i(\succsim)$ and $\omega_i \succ_j \varphi_j(\succsim)$, which contradicts Lemma 1. \square

Proof of Theorem 4. By *individual rationality*, for each $\succsim \in \mathcal{D}^\Pi$, $\varphi(\succsim) \in \mathcal{I}(\succsim)$. We prove (ii) of the definition of sequential maximizing rules by a series of steps.

Step 1: $t = 1$ and $t' = 1$. Let $\succsim \in \mathcal{D}^\Pi$. Assume to the contrary that there is a pairwise exchange rule φ that satisfies *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness* but there are $\succsim \in \mathcal{D}^\Pi$ and $a \in \mathcal{I}(\succsim)$ such that $\#M_{1,1}(a) > \#M_{1,1}(\varphi(\succsim))$. (By definition, for each $a' \in \mathcal{A}^2$, $P_{1,1}(a') = \emptyset$.) Without any loss of generality, there exist a set $T \subset N$ and $h, h' \in N \setminus T$ such that for each $i \in T \cup \{h, h'\}$, $\omega_i \in \Pi(1)$, $\varphi_h(\succsim) \notin \Pi(1)$, $\varphi_{h'}(\succsim) \notin \Pi(1)$ and:

- (i) For each $i \in T \cup \{h, h'\}$, there exists $i' \in (T \cup \{h, h'\}) \setminus \{i\}$ such that $a_i = \omega_{i'}$.
- (ii) For each $j \in T$, there exists $j' \in T \setminus \{j\}$ such that $\varphi_j(\succsim) = \omega_{j'}$.

There are two cases.

Case i). $T = \emptyset$. Clearly, $a_h = \omega_{h'}$ and $a_{h'} = \omega_h$. Since $a \in \mathcal{I}(\succsim)$, $\omega_h \in D(\succsim_{h'})$ and $\omega_{h'} \in D(\succsim_h)$. Because $\varphi_h(\succsim) \notin \Pi(1)$ and $\varphi_{h'}(\succsim) \notin \Pi(1)$, this contradicts Lemma 1.

Case ii). $T \neq \emptyset$. Let $\succsim' \in \mathcal{D}^\Pi$ be such that for each $i \in T \cup \{h, h'\}$, $\succsim'_i = \succsim_i$, and for each $j \notin T \cup \{h, h'\}$, $D(\succsim'_j) = \{\varphi_j(\succsim)\}$. Let $j \notin T \cup \{h, h'\}$. By *individual rationality*, $\varphi_j(\succsim'_j, \succsim_{-j}) \in \{\omega_j, \varphi_j(\succsim)\}$. By *strategy-proofness*, $\varphi_j(\succsim'_j, \succsim_{-j}) \succsim'_j \varphi_i(\succsim)$. Then, $\varphi_j(\succsim'_j, \succsim_{-j}) = \varphi_j(\succsim)$ and by *non-bossiness*, $\varphi(\succsim'_j, \succsim_{-j}) = \varphi(\succsim)$. Repeating the same argument exchanging the preference of each patient, we obtain $\varphi(\succsim') = \varphi(\succsim)$. Let $\succsim'' \in \mathcal{D}^\Pi$ be such that $D(\succsim''_h) = \{a_h\} \in \Pi(1)$. By *individual rationality*, $\varphi_h(\succsim''_h, \succsim'_{-h}) \in \{\omega_h, a_h\}$. By *strategy-proofness*, $\varphi_h(\succsim') \succsim'_h \varphi_h(\succsim''_h, \succsim'_{-h})$. Hence, $\varphi_h(\succsim''_h, \succsim'_{-h}) = \omega_h$. By the definition of $(\succsim''_h, \succsim'_{-h})$ and *individual rationality*, for each $j \notin T \cup \{h, h'\}$, $\varphi_j(\succsim''_h, \succsim'_{-h}) \in \{\omega_j, \varphi_j(\succsim)\}$. By *2-efficiency*, for each $j \notin T \cup \{h, h'\}$, $\varphi_j(\succsim''_h, \succsim'_{-h}) = \varphi_j(\succsim)$. On the other hand, for each $j' \in T \cup \{h'\}$,

either there is $i' \in T \cup \{h'\}$ such that $\varphi_{j'}(\succ''_h, \succ'_{-h}) = \omega_{i'}$, or $\varphi_{j'}(\succ''_h, \succ'_{-h}) = \omega_{j'}$. Moreover, since $\varphi(\succ''_h, \succ'_{-h}) \in \mathcal{A}^2$ and $\varphi_h(\succ''_h, \succ'_{-h}) = \omega_h$, there is $h'' \in T \cup \{h'\}$ such that $\varphi_{h''}(\succ''_h, \succ'_{-h}) = \omega_{h''}$. Let $b \in \mathcal{A}^2$ be such that for each $i \in T \cup \{h, h'\}$, $b_i = a_i$ and for each $j \notin T \cup \{h, h'\}$, $b_j = \varphi_j(\succ)$. Note that for each $i \in T \cup \{h, h'\}$, $b_i \in \Pi(1)$. Then, for each $i' \in N \setminus \{h, h''\}$, $b_{i'} \succ'_{i'} \varphi_{i'}(\succ''_h, \succ'_{-h})$, $b_h \succ''_h \varphi_h(\succ''_h, \succ'_{-h})$, and $b_{h''} \succ'_{h''} \varphi_{h''}(\succ''_h, \succ'_{-h})$, which contradicts 2-efficiency.

Step 2: $t = 1$, $t' = 2$. The result for $t = 1$ and $t' = 2, \dots, m$ follows from slightly modified arguments. Assume to the contrary that there is a pairwise exchange rule φ that satisfies *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*, but there are $\succ \in \mathcal{D}^\Pi$ and $a \in \mathcal{I}(\succ)$ such that for each $i \in P_{1,2}(\varphi(\succ))$, $\varphi_i(\succ) = a_i$ but

$$\#M_{1,2}(a) > \#M_{1,2}(\varphi(\succ)).$$

Then, there are a set $T \subset N \setminus P_{1,2}(\varphi(\succ))$ and a pair of patients $h, h' \in N \setminus T$ such that for each $i \in T \cup \{h, h'\}$ $\omega_i \in (\Pi(1) \cup \Pi(2))$, $\omega_h \in \Pi(1)$, $\omega_{h'} \in \Pi(2)$, $\varphi_h(\succ) \notin (\Pi(1) \cup \Pi(2))$, $\varphi_{h'}(\succ) \notin \Pi(1)$, and

- (i) For each $i \in T$ with $\omega_i \in \Pi(1)$ there exists $i' \in T$ with $\omega_{i'} \in \Pi(2)$ such that $\varphi_i(\succ) = \omega_{i'}$.
- (ii) For each $j \in T \cup \{h, h'\}$ with $\omega_j \in \Pi(1)$, there is $j' \in T \cup \{h, h'\}$ with $\omega_{h'} \in \Pi(2)$ such that $a_j = \omega_{j'}$.

Note that $h, h' \notin P_{1,2}(\varphi(\succ))$. There are two cases.

Case i). $T = \emptyset$, with the arguments in the proof for $t = 1$ and $t' = 1$, we obtain a contradiction with Lemma 1.

Case ii). $T \neq \emptyset$. Let $\succ' \in \mathcal{D}^\Pi$ be such that for each $i \in T \cup \{h, h'\}$, $\succ'_i = \succ_i$ and for each $j \notin T \cup \{h, h'\}$, $D(\succ_j) = \{\varphi_j(\succ)\}$. By *individual rationality*, $\varphi_j(\succ'_j, \succ_{-j}) \in \{\omega_j, \varphi_j(\succ)\}$. By *strategy-proofness*, $\varphi_j(\succ'_j, \succ_{-j}) \succ'_j \varphi_i(\succ)$. Then, $\varphi_j(\succ'_j, \succ_{-j}) = \varphi_j(\succ)$ and by *non-bossiness*, $\varphi(\succ'_j, \succ_{-j}) = \varphi(\succ)$. Repeating the same argument exchanging the preference of each patient, we obtain $\varphi(\succ') = \varphi(\succ)$. Let $\succ''_{h'} \in \mathcal{D}^\Pi_{h'}$ be such that $D(\succ''_{h'}) = \{a_{h'}\} \in \Pi(1)$. By *individual rationality*, $\varphi_{h'}(\succ''_{h'}, \succ'_{-h'}) \in \{\omega_{h'}, a_{h'}\}$. By

strategy-proofness, $\varphi_{h'}(\tilde{z}') \succsim'_{h'} \varphi_{h'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'})$. Hence, $\varphi_{h'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \omega_{h'}$. By the definition of $(\tilde{z}''_{h'}, \tilde{z}'_{-h'})$ and by *individual rationality*, for each $j \notin T \cup \{h, h'\}$, $\varphi_j(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) \in \{\omega_j, \varphi_j(\tilde{z})\}$. By *2-efficiency*, for each $j \notin T \cup \{h, h'\}$, $\varphi_j(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \varphi_j(\tilde{z})$. On the other hand, for each $j' \in T \cup \{h\}$, either there is $i' \in T \cup \{h\}$ such that $\varphi_{j'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \omega_{i'}$, or $\varphi_{j'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \omega_{j'}$. Since $\varphi(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) \in \mathcal{A}^2$ and $\varphi_{h'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \omega_{h'}$, there is $h'' \in T \cup \{h\}$ such that $\varphi_{h''}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \omega_{h''}$. Note that since for each $i \neq h'$, $D(\tilde{z}'_i) \subseteq D(\tilde{z}_i)$ and $D(\tilde{z}''_{h'}) \subseteq (\tilde{z}_{h'}, \tilde{z}'_{-h'})$, and φ satisfies *individual rationality*, $\varphi(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) \in \mathcal{I}(\tilde{z})$. Since for each $a \in \mathcal{I}(\tilde{z})$, $\#M_{1,1}(\varphi(\tilde{z})) \geq \#M_{1,1}(a)$, we have that for each $j' \in T \cup \{h\}$ with $\omega_{j'} \in \Pi(1)$, $\varphi_{j'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) \in \Pi(2) \cup \{\omega_{j'}\}$. Let $b \in \mathcal{A}^2$ be such that for each $i \in T \cup \{h, h'\}$, $b_i = a_i$ and for each $j \notin T \cup \{h, h'\}$, $b_j = \varphi_j(\tilde{z}''_{h'}, \tilde{z}'_{-h'}) = \varphi_j(\tilde{z})$. Note that for each $i \notin T \cup \{h, h'\}$ with $\omega_i \in \Pi(1)$, $b_i \in \Pi(2)$ and for each $i' \in T \cup \{h, h'\}$ with $\omega_{i'} \in \Pi(2)$, $b_{i'} \in \Pi(1)$. Then, for each $j \in N \setminus \{h', h''\}$, $b_j \succsim'_j \varphi_j(\tilde{z}''_{h'}, \tilde{z}'_{-h'})$, $b_{h'} \succ''_{h'} \varphi_{h'}(\tilde{z}''_{h'}, \tilde{z}'_{-h'})$, and $b_{h''} \succ''_{h''} \varphi_{h''}(\tilde{z}''_{h'}, \tilde{z}'_{-h'})$, which contradicts *2-efficiency*.

In order to conclude the proof, we can apply iteratively the arguments in the proof of Step 2, to prove the result for $t = 1$ and $t' = 3, \dots, m$. Then, given that the result is true for $t = 1$, $t' = m$, the arguments in Step 2 directly apply to prove the result for $t = 2$ and $t' = 2$, and we can proceed iteratively with all the remaining steps till we reach $t = m$, $t' = m$. \square

References

- [1] J. Alcalde-Unzu and E. Molis. Exchange of indivisible goods and indifferences: Absorbing Sets mechanisms. *Games and Economic Behavior*, 73(1):1–16, 2011.
- [2] I. Ashlagi and A. E. Roth. New challenges in multi-hospital kidney exchange. *American Economic Review Papers and Proceedings*, (102 (3)):354–359, 2012.
- [3] I. Ashlagi and A. E. Roth. Free riding and participation in large scale, multi-hospital kidney exchange. Forthcoming, *Theoretical Economics*, 2013.

- [4] L. M. Ausubel and T. Morrill. Sequential kidney exchange, 2013. Forthcoming, *American Economic Journal: Microeconomics*, 2013
- [5] C. G. Bird. Group incentive compatibility in a market with indivisible goods. *Economics Letters*, 14(4):309 – 313, 1984.
- [6] A. Bogomolnaia and H. Moulin. A simple random assignment problem with a unique solution. *Economic Theory*, 19(3):623–636, 2002.
- [7] K. Chung. On the existence of stable roommate matchings. *Games and Economic Behavior*, 33:206–230, 2000.
- [8] F. L. Delmonico. Exchanging kidneys – advances in living-donor transplantation. *The New England Journal of Medicine*, 350:1812–1814, 2004.
- [9] F. L. Delmonico, G. S. Lipkowitz, P. E. Morrissey, J. S. Stoff, J. Himmelfarb, W. Harmon, M. Pavlakis, H. Mah, J. Goguen, R. Luskin, E. Milford, G. B. M. Chobanian, B. Bouthot, M. Lorber, and R. J. Rohrer. Donor kidney exchanges. *American Journal of Transplantation*, 4:1628–1634, 2004.
- [10] L. Ehlers. Coalitional strategy-proof house allocation. *Journal of Economic Theory*, 105(2):298 – 317, 2002.
- [11] S. E. Gentry, D. L. Segev, M. Simmerling, and R. A. Montgomery. Expanded kidney paired donation through participation by compatible pairs. *American Journal of Transplantation*, 7:99–107, 2007.
- [12] D. W. Gjertson. Explainable variation in renal transplant outcomes: A comparison of standard and expanded criteria donors. *Clinical Transplants*, 2004:303 – 314, 2004.
- [13] D. W. Gjertson and J. Cecka. Living unrelated donor kidney transplantation. *Kidney International*, 58:491–499, 2000.
- [14] J. W. Hatfield. Pairwise kidney exchange: Comment. *Journal of Economic Theory*, 125:189–193, 2005.

- [15] A. Hylland and R. Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political Economy*, 87:293–314, 1979.
- [16] P. Jaramillo and V. Manjunath. The difference indifference makes in strategy-proof allocation of objects. *Journal of Economic Theory*, 147:1913–1946, 2012.
- [17] K. Keizer, M. de Klerk, B. Haase-Kromwijk, and W. Weimar. The dutch algorithm for allocation in living donor kidney exchange. *Transplantation Proceedings*, 37:589–591, 2005.
- [18] J. Ma. Strategy-proofness and the strict core in a market with indivisibilities. *International Journal of Game Theory*, 23:75–83, 1994.
- [19] E. Miyagawa. Strategy-proofness and the core in house allocation problems. *Games and Economic Behavior*, 38(2):347 – 361, 2002.
- [20] NHS Blood and Transplant. National matching scheme for paired and pooled (kidney) donation, 2009. <http://www.organdonation.nhs.uk>
- [21] A. Nicolò and C. Rodríguez-Álvarez. Transplant quality and patients’ preferences in paired kidney exchange. *Games and Economic Behavior*, 74(1):299–310, 2012.
- [22] C. M. Øien, A. V. Reisæter, T. Leivestad, F. W. Dekker, P. D. Line, and I. Os. Living donor kidney transplantation: The effects of donor age and gender on short- and long-term outcomes. *Transplantation*, 83(5):600 – 606, 2007.
- [23] G. Opelz. Impact of HLA compatibility on survival of kidney transplants from unrelated live donors. *Transplantation*, 64:1473–1475, 1997.
- [24] G. Opelz for the Collaborative Transplant Study. Hla compatibility and kidney grafts transplants from unrelated live donors. *Transplantation Proceedings*, 30:704–705, 1998.
- [25] Organización Nacional de Trasplantes. Programa nacional de donación renal cruzada en España. Technical report, Documentos de Consenso ONT, September 2009.

- [26] S. Pápai. Strategy-proof single unit award rules. *Social Choice and Welfare*, 18(4): 785–798, 2001.
- [27] K. Park, J. Lee, K. Huh, S. Kim, and Y. Kim. Exchange living-donor kidney transplantation: Diminution of donor organ shortage. *Transplantation Proceedings*, 36: 2949–2951, 2004.
- [28] M. A. Rees, J. E. Kopke, R. P. Pelletier, D. L. Segev, M. E. Rutter, A. J. Fabrega, J. Rogers, O. G. Pankewycz, J. Hiller, A. E. Roth, T. Sandholm, M. U. Ünver, and R. A. Montgomery. A nonsimultaneous, extended, altruistic–donor chain. *The New England Journal of Medicine*, 360:1096–1101, 2009.
- [29] A. E. Roth. Incentive compatibility in a market with indivisible goods. *Economics Letters*, 9:127–132, 1982.
- [30] A. E. Roth, T. Sönmez, and M. U. Ünver. Kidney exchange. *Quarterly Journal of Economics*, 119:457–488, 2004.
- [31] A. E. Roth, T. Sönmez, and M. U. Ünver. Pairwise kidney exchange. *Journal of Economic Theory*, 125:151–188, 2005.
- [32] A. E. Roth, T. Sönmez, and M. U. Ünver. A kidney exchange clearinghouse in New England. *American Economic Review, Papers and Proceedings*, 95:376–380, 2005.
- [33] A. E. Roth, T. Sönmez, M. U. Ünver, F. L. Delmonico, and S. L. Saidman. Utilizing list exchange and nondirected donations through “chain” paired kidney donations. *American Journal of Transplantation*, 6:2694–2705, 2006.
- [34] A. E. Roth, T. Sönmez, and M. U. Ünver. Efficient kidney exchange: Coincidence of wants in a markets with compatibility preferences. *American Economic Review*, 97–3:828–851, 2007.
- [35] L. Shapley and H. Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1:23–37, 1974.
- [36] T. Sönmez. Strategy-proofness and essentially single-valued cores. *Econometrica*, 67: 677–689, 1999.

- [37] T. Sönmez and M. U. Ünver. Altruistically unbalanced kidney exchange. Unpublished Manuscript, Boston College, 2013.
- [38] X. Su, S. A. Zenios, and G. M. Chertow. Incorporating recipient choice in kidney transplantation. *Journal of the American Society of Nephrology*, 15:1656–1663, 2004.
- [39] L.-G. Svensson. Strategy-proof allocation of indivisible goods. *Social Choice and Welfare*, 16(4):557–567, 1999.
- [40] M.U. Ünver. Dynamic kidney exchange. *Review of Economic Studies*, 77:372–414, 2010.
- [41] Ö. Yilmaz. Kidney exchange: An egalitarian mechanism. *Journal of Economic Theory*, 146:592–618, 2011.
- [42] Ö. Yilmaz. Kidney exchange: Further utilization of donors via listed exchange. Forthcoming *Journal of Mathematical Economics*, 2013.
- [43] S. A. Zenios. Optimal control of a paired-kidney exchange program. *Management Science*, 48:328–342, 2002.

Appendices. NOT FOR PUBLICATION

A Omitted Proofs

We start by providing precise statements of complementary results that are mentioned in the text.

Theorem 5. *Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be an age structure with $m \geq 2$. For each $k \in \mathbb{N}$ such that $3 \leq k \leq n-2$, no rule $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, strategy-proofness, and non-bossiness.*

Proof. If Π satisfies Assumption A, the result follows directly from Theorem 1 in the text. So assume to the contrary that $\Pi = \{\Pi(1), \Pi(2)\}$ does not satisfies assumption A and there is k ($3 \leq k \leq n-2$), and $\varphi : \mathcal{D}^\Pi \rightarrow \mathcal{A}^k$ such that φ satisfies the four properties. We have two cases:

- **Case i).** $\Pi(1) = \{\omega_1\}$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_1) = \{\omega_2, \dots, \omega_{k+2}\}$, for each $i = 2, \dots, k+1$, $D(\succsim_i) = \{\omega_1, \omega_{i+1}\}$, $D(\succsim_{k+2}) = \{\omega_1, \omega_2\}$. By *individual rationality*, we can on the assignment of kidneys restricted to the patients $\{\omega_1, \dots, \omega_{k+2}\}$. (For the sake of simplifying notation assume that $n > 5$. If $n = 5$, the argument applies directly by letting $k+2 = 5$.) By *k-efficiency*, $\varphi(\succsim) \in \mathcal{A}^k \setminus \mathcal{A}^{k-1}$. By construction, we can assume without loss of generality that

$$\varphi_i(\succsim) = \begin{cases} \omega_4 & \text{if } i = 1, \\ \omega_1 & \text{if } i = 2, \\ \omega_i & \text{if } i \in \{3, 4\}, \\ \omega_{i+1} & \text{if } 5 \leq i \leq k+1, \\ \omega_2 & \text{if } i = k+2. \end{cases}$$

Let $\succsim' \in \mathcal{D}^\Pi$ be such that for each $i \in \{2\} \cup \{5, \dots, k+1\}$, $D(\succsim'_i) = \varphi_i(\succsim)$ and for each $j \notin \{2\} \cup \{5, \dots, k+1\}$, $\succsim'_j = \succsim_j$. By *strategy-proofness*, $\varphi_2(\succsim'_2, \succsim_{-2}) \succsim'_2 \varphi_2(\succsim)$. Then, $\varphi_2(\succsim'_2, \succsim_{-2}) = \omega_1$. By *non-bossiness*, $\varphi(\succsim'_2, \succsim_{-2}) = \varphi(\succsim)$. Repeating the argument with the remaining patients, $\varphi(\succsim') = \varphi(\succsim)$. Let $\succsim'' \in \mathcal{D}^\Pi$ be such that for each $i \in \{3, 4\}$, $D(\succsim''_i) = \{\omega_1\}$ and for each $j \notin \{3, 4\}$, $\succsim''_j = \succsim'_j$. By *individual*

rationality, $\varphi_3(\tilde{\lambda}_3'', \tilde{\lambda}_{-3}') \in \{\omega_1, \omega_3\}$. By *strategy-proofness*, $\varphi_3(\tilde{\lambda}') \tilde{\lambda}_3' \varphi_3(\tilde{\lambda}_3'', \tilde{\lambda}_{-3}')$. Thus, $\varphi_3(\tilde{\lambda}_3'', \tilde{\lambda}_{-3}') = \varphi_3(\tilde{\lambda}') = \omega_3$. By *non-bossiness*, $\varphi(\tilde{\lambda}_3'', \tilde{\lambda}_{-3}') = \varphi(\tilde{\lambda}')$. Repeating the argument with patient 4, $\varphi(\tilde{\lambda}'') = \varphi(\tilde{\lambda}') = \varphi(\tilde{\lambda})$. Let $\tilde{\lambda}''' \in \mathcal{D}^\Pi$ be such that $D(\tilde{\lambda}_1) = \{\omega_5\}$, $D(\tilde{\lambda}_5) = \{\omega_1\}$, and for each $j \notin \{1, 5\}$, $\tilde{\lambda}_j''' = \tilde{\lambda}_j''$. By *individual rationality*, $\varphi_1(\tilde{\lambda}_1''', \tilde{\lambda}_{-1}'') \in \{\omega_1, \omega_5\}$. By *strategy-proofness*, $\varphi_1(\tilde{\lambda}_1''', \tilde{\lambda}_{-1}'') \tilde{\lambda}_1''' \varphi_1(\tilde{\lambda}'') = \omega_5$. Then $\varphi_1(\tilde{\lambda}_1''', \tilde{\lambda}_{-1}'') = \varphi_1(\tilde{\lambda}'')$, and by *non-bossiness*, $\varphi(\tilde{\lambda}_1''', \tilde{\lambda}_{-1}'') = \varphi(\tilde{\lambda}'')$. Finally, by *individual rationality*, $\varphi_5(\tilde{\lambda}''') \in \{\omega_1, \omega_5\}$. By *strategy-proofness*, $\omega_2 = \varphi_5(\tilde{\lambda}_1''', \tilde{\lambda}_{-1}'') \tilde{\lambda}_5'' \varphi_5(\tilde{\lambda}''')$. Then, $\varphi_5(\tilde{\lambda}''') = \omega_5$ and by *individual rationality*, $\varphi_1(\tilde{\lambda}''') = \omega_1$. Let $b \in \mathcal{A}^k$, be such that for each $j \notin \{1, 5\}$, $b = \varphi_j(\tilde{\lambda}''')$ and $b_1 = \omega_5$, $b_5 = \omega_1$. Note that for each $i \in N$, $b_i \tilde{\lambda}_i''' \varphi_i(\tilde{\lambda}''')$, and $b_1 \tilde{\lambda}_1''' \varphi_1(\tilde{\lambda}''')$ and $b_5 \tilde{\lambda}_5''' \varphi_5(\tilde{\lambda}''')$, which contradicts *k-efficiency*.

- **Case ii).** $\Pi(2) = \{\omega_n\}$. Let $\tilde{\lambda} \in \mathcal{D}^\Pi$ be such that for each $i \in \{1, \dots, k\}$, $D(\tilde{\lambda}) = \{\omega_{i+1}, \omega_n\}$, $D(\tilde{\lambda}_{k+1}) = \{\omega_1, \omega_n\}$, and $D(\tilde{\lambda}_n) = \{\omega_1, \dots, \omega_{k+1}\}$. By *individual rationality*, we can focus on the assignment of kidneys restricted to the patients $\{\omega_1, \dots, \omega_{k+1}\} \cup \{\omega_n\}$. By *k-efficiency*, $\varphi(\tilde{\lambda}) \in \mathcal{A}^k \setminus \mathcal{A}^{k-1}$. By construction, we can assume without loss of generality that

$$\varphi_i(\tilde{\lambda}) = \begin{cases} \omega_{i+1} & \text{if } i \in \{1, \dots, k-2\} \\ \omega_n & \text{if } i = k-1, \\ \omega_i & \text{if } i \in \{k, k+1\}, \\ \omega_1 & \text{if } i = n. \end{cases}$$

Let $\tilde{\lambda}' \in \mathcal{D}^\Pi$ be such that for each $i \in \{1, \dots, k-2\}$, $D(\tilde{\lambda}_i) = \varphi_i(\tilde{\lambda})$, and for each $j \geq k-1$, $\tilde{\lambda}_j' = \tilde{\lambda}_j$. By *strategy-proofness*, $\varphi_1(\tilde{\lambda}_1', \tilde{\lambda}_{-1}') \tilde{\lambda}_1' \varphi_1(\tilde{\lambda}) = \omega_2$. Then, $\varphi_1(\tilde{\lambda}_1', \tilde{\lambda}_{-1}') = \varphi_1(\tilde{\lambda})$, and by *non-bossiness*, $\varphi(\tilde{\lambda}_1', \tilde{\lambda}_{-1}') = \varphi(\tilde{\lambda})$. Repeating the argument with patients $i = 2, \dots, k-2$, one at a time, we obtain $\varphi(\tilde{\lambda}') = \varphi(\tilde{\lambda})$. Let $\tilde{\lambda}'' \in \mathcal{D}^\Pi$ be such that $D(\tilde{\lambda}_k'') = \{\omega_{k+1}\}$ and for each $j \neq k$, $\tilde{\lambda}_j'' = \tilde{\lambda}_j'$. By *individual rationality*, $\varphi_k(\tilde{\lambda}_k'', \tilde{\lambda}_{-k}') \in \{\omega_k, \omega_{k+1}\}$. By *strategy-proofness*, $\varphi_k(\tilde{\lambda}') \tilde{\lambda}_k' \varphi_k(\tilde{\lambda}_k'', \tilde{\lambda}_{-k}')$. Hence, $\varphi_k(\tilde{\lambda}_k'', \tilde{\lambda}_{-k}') = \omega_k = \varphi_k(\tilde{\lambda}')$ and by *non-bossiness*, $\varphi_k(\tilde{\lambda}_k'', \tilde{\lambda}_{-k}') = \varphi_k(\tilde{\lambda}')$ and $\varphi(\tilde{\lambda}'') = \varphi(\tilde{\lambda}')$. To conclude, let $\tilde{\lambda}''' \in \mathcal{D}^\Pi$ be such that $D(\tilde{\lambda}_{k-1}''') = \{\omega_k\}$ and for each $j \neq k-1$, $\tilde{\lambda}_j''' = \tilde{\lambda}_j''$. By *individual rationality*, $\varphi_{k-1} \in \{\omega_{k-1}, \omega_k\}$. By

k -efficiency,

$$\varphi_i(\succsim''') = \begin{cases} \omega_i & \text{if } i \in \{1, 2\}, \\ \omega_{i+1} & \text{if } i \in \{3, \dots, k\}, \\ \omega_n & \text{if } i = k + 1, \\ \omega_3 & \text{if } i = n. \end{cases}$$

Note that $\varphi_{k-1}(\succsim''') \succ_{k-1}'' \varphi_{k-1}(\succsim'')$ which contradicts *strategy-proofness*.

□

Weak Coalitional Strategy-Proofness. For each $\succsim \in \mathcal{D}$, there does not exist $T \subseteq N$ and $\succsim'_T \in \mathcal{D}_T$ such that for each $i \in T$, $\varphi_i(\succsim'_T, \succsim_{N \setminus T}) \succ_i \varphi_i(\succsim)$.

Theorem 6. For each age structure Π and each priority ordering σ , if σ respects Π then the age-based priority rule ψ^σ satisfies weak coalitional strategy-proofness in \mathcal{D}^Π .

Proof. In order to simplify notation, we consider the natural priority ordering σ^* . The arguments apply directly for every arbitrary priority ordering that respects Π . Assume to the contrary that ψ^* violates *weak coalitional strategy-proofness*. Then, there exists $T \subseteq N$, $\succsim \in \mathcal{D}^\Pi$, $\succsim'_T \in \times_{i \in T} \mathcal{D}_i^\Pi$ such that for each $i \in T$, $\psi_i^*(\succsim'_T, \succsim_{N \setminus T}) \succ_i \psi_i^*(\succsim)$. Let patient $j \in T$ be such that for each $k \in T$, $j \leq k$. Since $j \in T$, $\succsim_j \neq \succsim'_j$ and $\psi_j^*(\succsim'_T, \succsim_{N \setminus T}) \succ_j \psi_j^*(\succsim)$, necessarily $\psi_j^*(\succsim'_T, \succsim_{N \setminus T}) \neq \psi_j^*(\succsim)$. There are two possibilities:

Case i). For each $i < j$, $\psi_i^*(\succsim'_T, \succsim_{N \setminus T}) = \psi_i^*(\succsim)$ and for some patient k , $\omega_k = \psi_j^*(\succsim'_T, \succsim_{N \setminus T})$.

By the definition of ψ^* , $\omega_j \in D(\succsim'_k) \setminus D(\succsim_k)$ and $\omega_k \succ_k \omega_j$. Because $\succsim_k \neq \succsim'_k$, $k \in T$.

However, by ψ^* 's *individual rationality*, $\psi_k^*(\succsim) \succsim_k \omega_k \succ_k \omega_j = \psi_k^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$.

Case ii). There exists $i < j$ such that $\psi_i^*(\succsim'_T, \succsim_{N \setminus T}) \neq \psi_i^*(\succsim)$. Let $i' < j$ be such that for each $i < i'$ $\psi_i^*(\succsim'_T, \succsim_{N \setminus T}) = \psi_i^*(\succsim)$. If $\psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T}) \succ_{i'} \psi_{i'}^*(\succsim)$, by the definition of age based priority rule, there exists $k \in T$ such that $\omega_{i'} \notin D(\succsim_k)$ and $\omega_{i'} \in D(\succsim'_k)$ and $\psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T}) = \omega_k$. However, by *individual rationality*, $\psi_k^*(\succsim) \succsim_k \omega_k \succ_k \omega_{i'} = \psi_k^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$. Finally, if $\psi_{i'}^*(\succsim) \succ_{i'} \psi_{i'}^*(\succsim'_T, \succsim_{N \setminus T})$, then there exists $k' \in T$ such that $\psi_{i'}^*(\succsim) = \omega_{k'}$. By the definitions of ψ^* and i' ,

$\omega_{i'} \in D(\succsim_{k'}) \setminus D(\succsim'_{k'})$, and for each $\succsim_{k'} \in \mathcal{D}^\Pi$, $\omega_{i'} = \psi_{k'}^*(\succsim) \succsim_{k'} \psi_{k'}^*(\succsim'_T, \succsim_{N \setminus T})$, which contradicts $k \in T$.

Because both cases exhaust all the possibilities, this suffices to prove *weak coalitional strategy-proofness*. \square

B Generalized Π Based Preferences

Let $\mathcal{C} = \{\Pi^1, \dots, \Pi^S\}$ denote a collection of partitions of Ω . For each $s \in \{1, \dots, S\}$, $\Pi^s = \{\Pi^s(1), \dots, \Pi^s(m^s)\}$ is partition of Ω according to **characteristic** s . We write $\omega \succ^s \omega'$ if and only if $\omega \in \Pi^s(l)$ and $\omega' \in \Pi^s(l')$ with $m' > m$. We write $\omega' \geq^c \omega$ if and only if for all $s \leq S$, $\omega \geq^s \omega'$ and $\exists s' \leq S$ such that $\omega \succ^{s'} \omega'$. We write $\omega' =^c \omega$ if and only if for each $s \leq S$ and $l^s \leq m^s$, $\omega \in \Pi^s(l^s)$ implies $\omega' \in \Pi^s(l^s)$.

For each patient $i \in N$, the preference relation $\succsim_i \in \mathcal{P}$ is a **generalized \mathcal{C} -based preference** if for each $\omega, \omega' \in D(\succsim_i)$ and for each $\bar{\omega} \in ND(\succsim_i)$:

1. if $\omega' \geq^c \omega$ then $\omega \succ_i \omega'$;
2. if $\omega' =^c \omega$ then $\omega \sim_i \omega'$.
3. $\omega_i \succ_i \bar{\omega}$.

Let \mathcal{D}_i^c denote the set of all \mathcal{C} -based preferences for patient i and let $\mathcal{D}^c \equiv \times_{i \in N} \mathcal{D}_i^c$.

According to \mathcal{C} -based preferences, for each characteristic s (age, health status, etc.) Π^s partitions (in decreasing order) the set of available kidneys in subsets of kidneys which are homogeneous according to characteristic s . Thus, if Π^1 is the partition of available kidneys generated according to the characteristic $s = 1$, say age, $\Pi^1(1)$ contains the youngest (and therefore best according to this characteristic) kidneys, $\Pi^1(2)$ the second youngest kidneys, and $\Pi^1(m^1)$ the oldest kidneys. Condition 1 says that for each patient i , if a compatible kidney ω is better ranked than kidney ω' according to one characteristic and no worst according to each other characteristic, then kidney ω is preferred to the latter ω' . Condition 2 says that for each patient i if a kidney ω belongs the the same group as another kidney ω' according to each characteristic s , then the two kidneys are

indifferent for patient i . The last Condition 3 simply says that for each patient i , every desirable kidney is strictly preferred to every undesirable kidney.

The three conditions do not impose any constraint for each pair of compatible kidneys $\omega, \tilde{\omega}$ such that there exist two characteristics s, s' with $\omega >^s \tilde{\omega}$ and $\omega <^{s'} \tilde{\omega}$. According to a **generalized \mathcal{C} -based preference relation**, even if patients agree on how to rank every kidney according to each characteristic, they may not agree on the relevance of the different characteristics for the overall evaluation of the kidneys. Even if all patients prefer *ceteris paribus*, a younger kidney to an older one, a patient may prefer a younger kidney from a diabetic donor to an older kidney from an healthier donor while another patient may prefer the latter kidney to the former one. It is straightforward to note that if $S = 1$, then the generalized \mathcal{C} -based preference and our original Π age based preference coincide.

We consider the above preference domain a natural generalization in an ordinal framework of the age based preference domain. Unfortunately, the next proposition proves that if there are more than one relevant characteristic and at least two pairs of kidneys that are ranked differently according to two characteristics, then there is no rule satisfying *individual rationality*, *k-efficiency*, and *strategy-proofness*.

Assumption B. For each collection of partitions $\mathcal{C} = \{\Pi^1, \dots, \Pi^S\}$ with $\#S \geq 2$, there are $s, s' \leq S$ and $i, i', i'' \in N$ such that $\Pi^s \geq 3$, $\omega_i >^s \omega_{i'} >^s \omega_{i''}$, and $\omega_{i''} >^{s'} \omega_i$.

Proposition 3. Let $\mathcal{C} = \{\Pi^1, \dots, \Pi^S\}$ be a collection of partitions that satisfies Assumption B. For each $2 \leq k \leq n - 1$, there is no rule $\varphi : \mathcal{D}^{\mathcal{C}} \rightarrow \mathcal{A}^k$ such that φ satisfies individual rationality, *k-efficiency*, and strategy-proofness.

Proof. Since Assumption B implies Assumption A, by Theorem 1, we have only to prove that no pairwise exchange rule satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness*. Without loss of generality, we focus on $N = \{1, 2, 3\}$ (by *individual rationality*, assume all other patients are incompatible with these three patients). Let the collection of partitions \mathcal{C} be such that there are partitions $\Pi^s, \Pi^{s'} \in \mathcal{C}$ such that $\omega_1 >^s \omega_2 >^s \omega_3$ and $\omega_3^{s'} > \omega_1$. Let $\succsim \in \mathcal{D}^{\mathcal{C}}$ be such that for each patient $i \in \{1, 2, 3\}$ and $D(\succsim_i) = \{\omega_j \mid j \in \{1, 2, 3\} \setminus \{i\}\}$, $\omega_2 \succ_1 \omega_3$, $\omega_3 \succ_2 \omega_1$, and $\omega_1 \succ_3 \omega_2$. To interpret this preference profile

assume for instance that patients 1 and 3 care more about characteristic s than about characteristic s' , patient 2 cares more about the characteristic s' than about characteristic s , and patient 3 cares about characteristic s . The proof follows immediately applying the first part of the proof of Theorem 1 in Nicolò and Rodríguez-Álvarez [21]. \square

C Public Information about Compatibility

In this section we briefly discuss the alternative framework in which compatibilities among donors and patients are public information. For each patient i let $\mathbf{C}_i \subseteq \Omega$ be i 's set of compatible kidneys. That is, the kidneys that according to the medical test carried by the transplant coordinator will not be rejected by i 's immune system and that may result in a successful transplantation. Let $\mathbf{C} \equiv \{\mathbf{C}_i\}_{i \in N} \in \Omega^n$ denote a **compatibility profile**. If the patients are allowed to refuse any compatible kidney, we could define the sets of desirable (and undesirable kidneys) just as those compatible kidneys that are preferred to each patient's donor kidney. With such definition, all our previous results would immediately follow.²³ When the compatibility information is publicly observable, however, the arbitrary refusal of compatible kidneys does not seem adequate in the age-based environment. Following Nicolò and Rodríguez-Álvarez [21], it seems reasonable to assume that whenever a patient considers a compatible kidney as desirable, then she also considers all the compatible kidneys from younger donors as desirable. In this framework, the only relevant information that remains private is the minimal quality (maximal age) that a patient requires to undergo transplantation. This observation leads to an additional restriction of patients' preferences.

Let $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ be an age structure and \mathbf{C} a compatibility profile.

The preference relation $\succsim_i \in \mathcal{P}$ is a Π **age based preference consistent with \mathbf{C}** if

- (i) for each $\omega \in D(\succsim_i)$ with $\omega \in \Pi(m)$, and for each $\omega' \in \mathbf{C}_i \setminus \{\omega_i\}$ such that $\omega' \in \Pi(m')$ and $m' \leq m$, $\omega' \in D(\succsim_i)$.

²³Arbitrary refusals may be originated by any kind of preconception, or they may incorporate at some extent the preferences of the donor. For instance, the donor may accept the non-related living donation, if no long travels are required.

- (ii) $\omega_i \in \mathbf{C}_i$ and $\omega_i \in \Pi(j)$ imply that for each $\omega' \in \Pi(j')$ with $j < j'$, $\omega' \in ND(\succsim_i)$.
- (iii) for each $\omega, \omega' \in D(\succsim_i)$, $\omega \in \Pi(j)$ and $\omega' \in \Pi(k)$, $\omega \succsim_i \omega'$ if and only if $j \leq k$.
- (iv) for each $\bar{\omega}, \bar{\omega}' \in ND(\succsim_i)$, $\omega_i \succ_i \bar{\omega}$ and $\bar{\omega} \sim_i \bar{\omega}'$.

We denote by $\mathcal{D}_i^\Pi(\mathbf{C})$ the domain of Π age based preferences consistent with \mathbf{C} for patient i and $\mathcal{D}^\Pi(\mathbf{C}) \equiv \times_i \mathcal{D}_i^\Pi(\mathbf{C})$.

If \mathbf{C} is public information, the ranking of desirable kidneys is determined by Π , and each patient's set of desirable kidneys depends both on Π and \mathbf{C} . By (i), if a compatible kidney is desirable for patient i , then all the compatible kidneys from a donor with lower (or equal) age are also desirable. By (ii), if patient i 's donor is compatible with patient i , then the kidneys from compatible donors older than patient i 's donor are not desirable. Items (iii) and (iv) just reproduce the notions of age-based preferences presented in section 3. This preference domain is similar to the domain proposed by Bogomolnaia and Moulin [6] in a problem of random house allocation. These authors assume that there is a common (strict) ranking of objects but there is no restriction on how each agent ranks her outside option (not receiving any object) with respect to the available objects.

Note that if the information about compatibility is public, the only information about each patient's preference that remains private is the older compatible kidney that she is willing to receive. Note that our definition incorporates the possibility of altruistic motivations. A patient may have a compatible donor, but she may prefer to receive compatible kidneys that belong to the same element of the age structure Π . (See Sönmez and Ünver [37].) Alternatively, if she prefers her donor's kidney to compatible kidneys in the same element of the partition, the patient accepts an exchange only if she improves upon her donor's kidney. For each patient i , each age structure Π , and each \mathbf{C} , $\mathcal{D}_i^\Pi(\mathbf{C}) \subset \mathcal{D}_i^\Pi$. On the other hand, the transplant coordinator may use the information in \mathbf{C} in the definition of the rule. Hence, we deal with \mathbf{C} specific rules, $\varphi^\mathbf{C} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^k$. With the introduction of new notation, we can now state new versions of our results for specific \mathbf{C} .

Theorem 7. *For each age structure Π , each $k \in \mathbb{N}$ such that $3 \leq k \leq n - 1$, there are \mathbf{C} no rule $\varphi^\mathbf{C} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, and strategy-proofness.*

Proof. The arguments in the proof of Theorem 1 apply directly. For each patient i , let $\mathbf{C}_i = \{\omega_{i+1}, \omega_{i+2}\}$ (modulo n). Note that all the preferences we used in the proof of Theorem 1 belong to $\mathcal{D}^\Pi(\mathbf{C})$. \square

Since for each \mathbf{C} , $\mathcal{D}_i^\Pi(\mathbf{C}) \subset \mathcal{D}_i^\Pi$, Proposition 1 applies and under the age structure Π^* , the *age-based priority rule* always selects the unique core-stable assignment.

Theorem 8. *For each \mathbf{C} , a rule $\varphi^\mathbf{C} : \mathcal{D}^{\Pi^*}(\mathbf{C}) \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency and strategy-proofness if and only if $\varphi^\mathbf{C}$ is the age-based priority rule ψ^* .*

Proof. The arguments in the proof of Theorem 2 apply directly once we observe that the domain $\mathcal{D}^{\Pi^*}(\mathbf{C})$ satisfies Sönmez [36]’s conditions. \square

With Theorem 8 at hand and with the fact that $\mathcal{D}_i^\Pi(\mathbf{C}) \subset \mathcal{D}_i^\Pi$, we obtain the following result.

Theorem 9. *For each age structure Π , each priority ordering σ , the priority rule ψ^σ satisfies strategy-proofness in $\mathcal{D}^\Pi(\mathbf{C})$ for each \mathbf{C} if and only if ψ^σ is an age-based priority rule.*

Proof. Consider an age structure Π and a priority ordering σ that does not respect Π . Hence, by Remark 2, there are $i, j, m \in N$, $k, k' \in \mathbb{N}$ be such that $\omega_i \in \Pi(k)$, $\omega_j \in \Pi(k')$, $k \leq k'$, $\sigma(i) > \sigma(j)$, and $\sigma_m > \sigma_j$. Let \mathbf{C} and $\succsim \in \mathcal{D}^\Pi$ be such that $\mathbf{C}_i = D(\succsim_i) = \{\omega_m\}$, $\mathbf{C}_j = D(\succsim_j) = \{\omega_m\}$, and $\mathbf{C}_m = D(\succsim_m) = \{\omega_i, \omega_j\}$. Clearly, $\psi_m^\sigma(\succsim) = \omega_j$. Let $\succsim'_m \in \mathcal{D}_m^\Pi$ be such that $D(\succsim'_m) = \{\omega_i\}$. Then, $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) = \omega_i$, and $\psi_m^\sigma(\succsim'_m, \succsim_{-m}) \succsim_m \psi_m^\sigma(\succsim)$, which proves that ψ^σ violates *strategy-proofness*. Since $\mathcal{D}^\Pi(\mathbf{C}) \subset \mathcal{D}^\Pi$, the arguments in the proof of Theorem 3 prove that every age-based priority rule satisfies *strategy-proofness* in $\mathcal{D}^\Pi(\mathbf{C})$. \square

We conclude with the generalization of Lemma 1 and Example 1 to the new environment.

Lemma 3. *There are \mathbf{C} such that if $\varphi^\mathbf{C} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for each $i, j \in N$ and $\succsim \in \mathcal{D}^\Pi(\mathbf{C})$, $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ either $\varphi_i^\mathbf{C}(\succsim) \succsim_i \omega_j$ or $\varphi_j^\mathbf{C}(\succsim) \succsim_j \omega_i$ (or both).*

Proof. Assume there are $\tilde{\mathbf{C}}$, an age structure Π , and $\varphi^{\tilde{\mathbf{C}}} : \mathcal{D}^{\Pi}(\tilde{\mathbf{C}})$ that satisfies *individual rationality*, *2-efficiency*, *strategy-proofness*, and *non-bossiness*. Take $i, j \in N$ and $\tilde{\succsim} \in \mathcal{D}^{\Pi}(\tilde{\mathbf{C}})$ such that $\omega_i \in D(\tilde{\succsim}_j)$ and $\omega_j \in D(\tilde{\succsim}_i)$ but $\omega_i \tilde{\succsim}_j \varphi_j^{\tilde{\mathbf{C}}}(\tilde{\succsim})$ and $\omega_j \tilde{\succsim}_i \varphi_i^{\tilde{\mathbf{C}}}(\tilde{\succsim})$. Define \mathbf{C} in such a way that $\mathbf{C}_i = \{\omega_j, \varphi_i^{\tilde{\mathbf{C}}}(\tilde{\succsim})\}$, $\mathbf{C}_j = \{\omega_i, \varphi_j^{\tilde{\mathbf{C}}}(\tilde{\succsim})\}$, and for each $k \in N \setminus \{i, j\}$, $\mathbf{C}_k = \varphi_k^{\tilde{\mathbf{C}}}(\tilde{\succsim})$. From now on, the proof literally replicates the arguments in the proof of Lemma 1. Assume to the contrary that there are $\varphi^{\mathbf{C}} : \mathcal{D}^{\Pi}(\mathbf{C}) \rightarrow \mathcal{A}^2$ and $\succsim \in \mathcal{D}^{\Pi}(\mathbf{C})$ such that $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ but $\omega_i \succsim_j \varphi_j^{\mathbf{C}}(\succsim)$ and $\omega_j \succsim_i \varphi_i^{\mathbf{C}}(\succsim)$. Let $\succsim' \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $\succsim'_i = \succsim_i$, $\succsim'_j = \succsim_j$, and for each $k \in N \setminus \{i, j\}$, $D(\succsim'_k) = \{\varphi_k^{\mathbf{C}}(\succsim)\}$. Let $k' \in N \setminus \{i, j\}$. By *individual rationality*, $\varphi_{k'}^{\mathbf{C}}(\succsim'_{k'}, \succsim'_{-k'}) \in \{\omega_{k'}, \varphi_{k'}^{\mathbf{C}}(\succsim)\}$. By *strategy-proofness*, $\varphi_{k'}^{\mathbf{C}}(\succsim'_{k'}, \succsim'_{-k'}) \succsim'_{k'} \varphi_{k'}^{\mathbf{C}}(\succsim)$. Hence, $\varphi_{k'}^{\mathbf{C}}(\succsim'_{k'}, \succsim'_{-k'}) = \varphi_{k'}^{\mathbf{C}}(\succsim)$, and by *non-bossiness*, $\varphi^{\mathbf{C}}(\succsim'_{k'}, \succsim'_{-k'}) = \varphi^{\mathbf{C}}(\succsim)$. Repeating the argument with the remaining patients (one at a time), we obtain $\varphi^{\mathbf{C}}(\succsim') = \varphi^{\mathbf{C}}(\succsim)$. Let $\succsim'' \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $D(\succsim''_i) = \{\omega_j, \varphi_i^{\mathbf{C}}(\succsim)\}$, $D(\succsim''_j) = \{\omega_i, \varphi_j^{\mathbf{C}}(\succsim)\}$ and for each patient $k \notin \{i, j\}$, $\succsim''_k = \succsim'_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i^{\mathbf{C}}(\succsim''_i, \succsim'_{-i}) = \varphi_i^{\mathbf{C}}(\succsim')$. By *non-bossiness*, $\varphi^{\mathbf{C}}(\succsim''_i, \succsim'_{-i}) = \varphi^{\mathbf{C}}(\succsim')$. Similarly, by *individual rationality* and *strategy-proofness*, $\varphi_j^{\mathbf{C}}(\succsim''_{\{i,j\}}, \succsim'_{-\{i,j\}}) = \varphi_j^{\mathbf{C}}(\succsim'_i, \succsim'_{-i})$. By *non-bossiness*, $\varphi^{\mathbf{C}}(\succsim''_{\{i,j\}}, \succsim'_{-\{i,j\}}) = \varphi^{\mathbf{C}}(\succsim'_i, \succsim'_{-i})$. Thus, $\varphi^{\mathbf{C}}(\succsim'') = \varphi^{\mathbf{C}}(\succsim')$. Let $\bar{\succsim} \in \mathcal{D}^{\Pi}(\mathbf{C})$ be such that $D(\bar{\succsim}_i) = \{\omega_j\}$, $D(\bar{\succsim}_j) = \{\omega_i\}$, and for each patient $k \notin \{i, j\}$, $\bar{\succsim}_k = \succsim''_k$. By *individual rationality* and *strategy-proofness*, $\varphi_i^{\mathbf{C}}(\bar{\succsim}_i, \bar{\succsim}_{-i}) = \{\omega_i\}$. By *individual rationality*, $\varphi_j^{\mathbf{C}}(\bar{\succsim}_i, \bar{\succsim}_{-i}) \in \{\omega_j, \varphi_j^{\mathbf{C}}(\bar{\succsim}'')\}$. By *2-efficiency*, $\varphi_j^{\mathbf{C}}(\bar{\succsim}_i, \bar{\succsim}_{-i}) = \varphi_j^{\mathbf{C}}(\bar{\succsim}'')$. Finally, by *individual rationality* and *strategy-proofness*, $\varphi_j^{\mathbf{C}}(\bar{\succsim}_{\{i,j\}}, \bar{\succsim}_{-\{i,j\}}) = \{\omega_j\}$ and $\varphi_i^{\mathbf{C}}(\bar{\succsim}_{\{i,j\}}, \bar{\succsim}_{-\{i,j\}}) = \{\omega_i\}$, which violates *2-efficiency*. \square

Example 2. Let $N = \{1, 2, 3, 4\}$, $\Pi(1) = \{\omega_1, \omega_2\}$, $\Pi(2) = \{\omega_3, \omega_4\}$, and \mathbf{C} such that:

$$\begin{aligned} \mathbf{C}_1 &= \{\omega_2, \omega_3, \omega_4\}, & \mathbf{C}_3 &= \{\omega_1\}, \\ \mathbf{C}_2 &= \{\omega_1, \omega_3, \omega_4\}, & \mathbf{C}_4 &= \{\omega_2\}. \end{aligned}$$

Consider the preference profile $\succsim \in \mathcal{D}^\Pi$ such that

| \succsim_1 | \succsim_2 | \succsim_3 | \succsim_4 |
|----------------------------|----------------------------|--------------|--------------|
| ω_2 | ω_1 | ω_1 | ω_2 |
| $\omega_3 \sim_1 \omega_4$ | $\omega_3 \sim_1 \omega_4$ | ω_3 | ω_4 |
| ω_1 | ω_2 | \dots | \dots |

Note that $(\omega_3, \omega_4, \omega_1, \omega_2) \in \mathcal{I}(\succsim)$. Replicating the arguments in the proofs of Lemmata 1 and 3, we obtain that for every rule $\varphi^{\mathbf{C}} : \mathcal{D}^\Pi(\mathbf{C}) \rightarrow \mathcal{A}^2$ that satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness,

$$\varphi^{\mathbf{C}}(\succsim) = (\omega_2, \omega_1, \omega_3, \omega_4).$$

Corollary 1. *There exists \mathbf{C} such that if a rule $\varphi^{\mathbf{C}}$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for some preference profile $\succsim \in \mathcal{D}^\Pi(\mathbf{C})$, $\varphi^{\mathbf{C}}(\succsim)$ does not maximize the number of mutually compatible kidney exchanges.*

D Multiple Donors

Sometimes patients in the waiting list may find more than one potential donor. If patients with multiple potential donors are keen to participate in PKE programs, algorithms have to take account of this aspect. Even if only one among the potential donors of a patient donates her kidney, the fact that a patient may have many potential donors can greatly increase the chances to find mutually compatible pairs. Since it is reasonable to assume that the information about how many potential donors a patient has is private information, rules should provide them incentives to reveal this valuable information. To analyze this general case, we need to slightly modify the framework and to incorporate some additional notation.

Let $N = \{i, \dots, n\}$ be a set of patients and $\Omega = \{\omega_1, \dots, \omega_{n'}\}$ be a set of available kidneys from living donors, $n \leq n'$.²⁴ For each patient i let Ω_i denote the set of kidneys from i 's donors. Clearly, $\cup_{i \in N} \Omega_i = \Omega$ and for each patient $j \neq i$, $\Omega_i \cap \Omega_j = \emptyset$. Let the mapping $\rho : N \rightarrow N$ be such that for each $j \leq n$ $\rho(j) = i$ if $\omega_j \in \Omega_i$.

²⁴Note that, from now on, kidneys' indexes do not refer to patients.

In the multiple donor case, we analyze patients' incentives to manipulate by reporting different sets of potential donors. The set of donors is an argument of the kidney assignment rule. For each patient i let \mathcal{K}_i be the set of non-empty subsets of Ω_i and $K_i \in \mathcal{K}_i$ be a set of donors that may be reported by patient i to the PKE program coordinator. Let $\mathcal{K} \equiv \times_{i \in N} \mathcal{K}_i$. We denote by $\mathbf{K} = (K_1, \dots, K_n)$ a generic element of \mathcal{K} . Let $\Omega = (\Omega_1, \dots, \Omega_n)$. For each $\mathbf{K} = (K_1, \dots, K_n) \in \mathcal{K}$ and for each $S \subseteq N$, \mathbf{K}_S denotes the restriction of \mathbf{K} to the members of S , and for each $\succsim \in \mathcal{P}$, $\succsim^{\mathbf{K}}$ is the restriction of \succsim to the kidneys $\cup_{i \in N} K_i$.

A **(generalized) assignment** a is an n -vector $a = (a_1, \dots, a_n)$, such that for each patient i $a_i \in \Omega$ and

- (i) for each $i, j \in N$, $i \neq j$ and each $\omega, \omega' \in \Omega$, if $a_i = \omega, a_j = \omega'$, then $\omega \neq \omega'$;
- (ii) for each $i \in N$, if for some $j \in N$, $a_j \in \Omega_i$, then for every $j' \neq j$, $a_{j'} \notin \Omega_i$.

We introduced this second requirement to convey the idea that for each patient at most one donor donates her kidney.

For each $\mathbf{K} \in \mathcal{K}$, we say that a generalized assignment a is *feasible* under the set of donors \mathbf{K} if for each $i, j \in N$, $a_i \in \Omega_j$ implies $a_i \in \mathbf{K}_j$. Let $\mathcal{A}(\mathbf{K})$ be the set of all *feasible* (generalized) assignments under \mathbf{K} , and for each $k \leq n$ let $\mathcal{A}^k(\mathbf{K})$ be the set of all (generalized) assignments with cardinality smaller than or equal to k that are *feasible* under \mathbf{K} .

A **generalized (kidney assignment) rule** is a mapping $\Phi : \mathcal{D} \times \mathcal{K} \rightarrow \mathcal{A}$ such that for each $\succsim \in \mathcal{D}$ and each $\mathbf{K} \in \mathcal{K}$, $\Phi(\succsim, \mathbf{K}) \in \mathcal{A}(\mathbf{K})$ and for each $\succsim, \succsim' \in \mathcal{D}$ and $\mathbf{K} \in \mathcal{K}$ such that $\succsim_{\mathbf{K}} = \succsim'_{\mathbf{K}}$, $\Phi(\succsim, \mathbf{K}) = \Phi(\succsim', \mathbf{K})$.

For each $\mathbf{K} \in \mathcal{K}$, the definitions of *k-efficiency*, *strategy-proofness*, and *non-bossiness* directly apply to the multiple donors scenario. *Individual rationality* may be immediately extended just by applying its logic to all the potential donors of each patient.

Individual Rationality for generalized rules. For each $i \in N$, each $\succsim \in \mathcal{D}$, each $\mathbf{K} \in \mathcal{K}$, and each $\omega \in \mathbf{K}_i$, $\Phi_i(\succsim, \mathbf{K}) \succsim_i \omega$.

In the multiple donors scenario, we assume that patients only care about the kidney they receive and do not have preferences over who will be the donor involved in the kidney exchange. Patients' preferences are still defined over all potential donors Ω . To conclude with the description of patients' preferences, we say that a kidney is desirable for patient i if it improves upon i 's all potential donors' kidneys. That is, for each patient i , i 's **set of desirable kidneys** is the set $D(\succsim_i) \equiv \{\omega \in \Omega \mid \forall \omega' \in \Omega_i, \omega \succsim_i \omega'\}$. With this definition of desirable kidneys, for each age structure $\Pi = \{\Pi(1), \dots, \Pi(m)\}$ of Ω , the notion of age based preferences in the multiple donors scenario simply replicates the definition in the single donor case. In order to save notation, we assume that for each Π and each $\succsim_i \in \mathcal{D}^\Pi$ there is a unique kidney in Ω_i , $\bar{\omega}(i, \succsim_i) = \{\omega \in \Omega_i \mid \text{for each } \omega' \in \Omega_i, \omega \succsim_i \omega'\}$.

Since the single donor framework is a special case of the multiple donors scenario, many important results of the single donor framework immediately extend to the multiple donors framework. Specifically, with the same arguments we use in the single donor case we obtain the following versions of Theorem 1, Lemma 1, and Theorem 4.

Theorem 10. *For each age structure Π , each $\mathbf{K} \in \mathcal{K}$, and each $k \in \mathbb{N}$ such that $3 \leq k \leq n - 1$, no generalized rule $\Phi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^k$ satisfies individual rationality, k -efficiency, and strategy-proofness.*

Lemma 4. *For each $\mathbf{K} \in \mathcal{K}$, if $\Phi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then for each $i, j \in N$, $\succsim_i \in \mathcal{D}^\Pi$, $\omega_i \in D(\succsim_j)$ and $\omega_j \in D(\succsim_i)$ imply either $\Phi_i(\succsim_i, \mathbf{K}) \succsim_i \omega_j$ or $\Phi_j(\succsim_j, \mathbf{K}) \succsim_j \omega_i$ (or both).*

Theorem 11. *For each $\mathbf{K} \in \mathcal{K}$, if $\Phi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ satisfies individual rationality, 2-efficiency, strategy-proofness, and non-bossiness, then φ is a sequential matching maximizing rule.*

However, we need to adopt the definition of age-based priority rules to the multiple donors case. Since patients may have donors whose kidneys belong to different classes of the age structure Π , the extension of the notion of *age-based priority rules* to the multiple donors case is not immediate. We devote this appendix to show the difficulties that arise in this setting and how to tailor age-based priority rules in order to preserve the properties that they satisfy in the single-donor scenario.

Example 3. Let $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \dots, \omega_5\}$, $\Pi = \{\Pi(1), \Pi(2)\}$ such that $\Pi(1) = \{\omega_1, \omega_2\}$, $\Pi(2) = \{\omega_3, \omega_4, \omega_5\}$ and $\Omega_1 = \{\omega_1, \omega_4\}$, $\Omega_2 = \{\omega_2\}$ and $\Omega_3 = \{\omega_3, \omega_5\}$. Let the generalized rule Φ be defined in such a way that, for each $\mathbf{K} \in \mathcal{K}$, $\Phi(\cdot, \mathbf{K})$ is a priority rule with priority ordering σ^* . The rule Φ assigns priority to patients with a donor in $\Pi(1)$. Let $\succsim \in \mathcal{D}^\Pi$ be such that $D(\succsim_1) = \{\omega_3\}$, $D(\succsim_2) = \{\omega_5\}$ and $D(\succsim_3) = \{\omega_2, \omega_4\}$. Let \mathbf{K} be such that $\mathbf{K} = \Omega$. Let $\succsim' \in \mathcal{D}^\Pi$ be such that $\succsim'_j = \succsim_j$ for $j = 1, 2$, and let $D(\succsim'_3) = \{\omega_2\}$. Clearly, $\Phi(\succsim, \mathbf{K}) = (\omega_3, \omega_2, \omega_4)$, $\Phi(\succsim', \mathbf{K}) = (\omega_1, \omega_5, \omega_2)$, and $\Phi_3(\succsim', \mathbf{K}) \succ_3 \Phi(\succsim, \mathbf{K})$. Thus, Φ violates strategy-proofness.

With multiple donors, it is necessary to define a multi-stage mechanism in order to maintain the non-manipulability of the age-based priority rule. At a first glance, Theorem 11 suggests that patients with young donors should have the right to choose first in the priority sequential procedure. It is not immediate, however, how to assign the priorities when patients may simultaneously have young and mature donors. To preserve *strategy-proofness*, it is necessary that younger kidneys are offered first.

Let us define the generalized priority algorithm based on natural order of the kidneys. For each $\succsim \in \mathcal{P}$ and each $\mathbf{K} \in \mathcal{K}$, let

$$\mathcal{GM}_0(\succsim, \mathbf{K}) = \mathcal{I}(\succsim, \mathbf{K}),$$

Next, for each $k = 1, \dots, n'$, define iteratively

$$\mathcal{S}_k(\succsim, \mathbf{K}) = \left\{ b \in \mathcal{GM}_{k-1}(\succsim, \mathbf{K}) \left| \begin{array}{ll} \text{either} & b_{\rho(k)} = \bar{\omega}(\rho(k), \succsim_{\rho(k)}) \\ \text{or} & b_{\rho(k)} \in \Omega_j \Rightarrow b_j \in \{\omega_1, \dots, \omega_k\} \cap \Omega_{\rho(k)} \end{array} \right. \right\}$$

and

$$\mathcal{GM}_k(\succsim, \mathbf{K}) = \{a \in \mathcal{GM}_{k-1}(\succsim, \mathbf{K}) \mid \text{for each } b \in \mathcal{S}_k(\succsim, \mathbf{K}), a_{\rho(k)} \succsim_{\rho(k)} b_{\rho(k)}\}.$$

For each $\mathbf{K} \in \mathcal{K}$ the generalized priority algorithm selects sequentially from the set of individually rational pairwise assignments. At stage 1, patient $\rho(1)$ considers the assignments in $\mathcal{I}(\succsim, \mathbf{K})$ that involve either not exchanging any kidney or exchanging kidney ω_1 , this is the set $\mathcal{S}_1(\succsim, \mathbf{K})$. Then, only those assignment in $\mathcal{I}(\succsim, \mathbf{K})$ that are at least as preferred as any assignment in $\mathcal{S}_1(\succsim, \mathbf{K})$ are selected to continue in the second stage. In a

nutshell, $\rho(1)$ leaves to the second stage those assignments that are at least as good as the best outcome $\rho(1)$ may obtain by exchanging ω_1 . At stage k , $\rho(k)$ leaves to stage $k + 1$ those assignments that are as good or improve upon the assignments $\rho(k)$ may obtain exchanging any $\omega' \in \{\omega_1, \dots, \omega_k\} \cap \Omega_{\rho(k)}$. Note that if $\omega_k \in \Omega_i$ and there is no $k' \geq k$ such that $\omega_{k'} \in \Omega_i$, $\rho(k) = i$ and for each $a, a' \in \mathcal{GM}_k(\succsim, \mathbf{K})$, $a_i \sim_i a'_i$. Hence, $\mathcal{GM}_n^\sigma(\succsim, \mathbf{K})$ is well defined, non-empty, and essentially single-valued.²⁵

A pairwise exchange rule $\Psi : \mathcal{D} \rightarrow \mathcal{A}^2$ is a **generalized age based priority rule** if for each $\succsim \in \mathcal{D}^\Pi$ and each \mathbf{K} , $\Psi(\succsim, \mathbf{K}) \in \mathcal{GM}_n(\succsim)$.

We focus first in the case of strict preferences over desirable kidneys. Hence, consider $\Pi^* = \{\Pi^*(1), \dots, \Pi^*(n')\}$, with $\Pi^*(l) = \{\omega_l\}$ for each $l = 1, \dots, n'$.

Proposition 4. *For each $\succsim \in \mathcal{D}^{\Pi^*}$, and each $\mathbf{K} \in \mathcal{K}$, $\mathcal{C}(\succsim, \mathbf{K}) = \{\Psi(\succsim, \mathbf{K})\}$.*

Proof. Let $\succsim \in \mathcal{D}^{\Pi^*}$. We first prove that for each $b \in \mathcal{A}^2 \setminus \Psi(\succsim, \mathbf{K})$, $b \notin \mathcal{C}(\succsim, \mathbf{K})$. If $b \notin \mathcal{I}(\succsim, \mathbf{K})$, then clearly $b \notin \mathcal{C}(\succsim)$. Let $b \in \mathcal{I}(\succsim, \mathbf{K})$. Let ω_i be the kidney such that $b_{\rho(i)} \neq \Psi_{\rho(i)}(\succsim, \mathbf{K})$ and for each $i' \in \mathbb{N}$ with $i' < i$, $b_{\rho(i')} = \Psi_{\rho(i')}(\succsim, \mathbf{K})$. Let ω_j be the kidney such that $\Psi_{\rho(i)}(\succsim, \mathbf{K}) = \omega_j$. Note that since $b \in \mathcal{I}(\succsim, \mathbf{K})$ and preferences over desirable kidneys are strict, by the definitions of Ψ and i , we have $i < j$ and $\Psi_{\rho(i)}(\succsim, \mathbf{K}) = \omega_j \succ_{\rho(i)} b_{\rho(i)}$. Because $\Psi_{\rho(j)}(\succsim, \mathbf{K}) = \omega_i \in D(\succsim_{\rho(j)})$, for each $i' \in \mathbb{N}$ such that $\omega_{i'} \succ_{\rho(j)} \omega_i$, $i' < i$. Thus, since for each $i' < i$, $\Psi_{\rho(i')}(\succsim, \mathbf{K}) = b_{\rho(i')} \neq \omega_j$, we have $\Psi_{\rho(j)}(\succsim, \mathbf{K}) = \omega_i \succ_{\rho(j)} b_{\rho(j)}$. Hence, the pair $\{\rho(i), \rho(j)\}$ weakly blocks b at \succsim via $\Psi(\succsim, \mathbf{K})$.

To conclude the proof, we show that $\Psi(\succsim, \mathbf{K}) \in \mathcal{C}(\succsim, \mathbf{K})$. Since $\Psi(\succsim, \mathbf{K}) \in \mathcal{I}(\succsim, \mathbf{K})$, there is no single-patient coalition that weakly blocks $\Psi(\succsim, \mathbf{K})$ at \succsim . Hence, we check that $\Psi(\succsim, \mathbf{K})$ is not weakly blocked by a pair of patients. Assume to the contrary that there are a pair $\{\omega_i, \omega_j\} \subset \Omega$ and $a \in \mathcal{A}^2$ such that $\{\rho(i), \rho(j)\}$ weakly block $\Psi(\succsim, \mathbf{K})$ at \succsim via a . Without loss of generality, let $i < j$. Since preferences over desirable kidneys are strict and $a \neq \Psi(\succsim, \mathbf{K})$, $a_{\rho(i)} = \omega_j \succ_{\rho(i)} \Psi_{\rho(i)}(\succsim, \mathbf{K})$ and $a_{\rho(j)} = \omega_i \succ_{\rho(j)} \Psi_{\rho(j)}(\succsim, \mathbf{K})$. Let $\omega_{j'}$ be the kidney such that $\Psi_{\rho(i)}(\succsim, \mathbf{K}) = \omega_{j'}$. There are two cases:

Case i). $j' < i$. Because $\succsim_{\rho(i)} \in \mathcal{D}_{\rho(i)}^{\Pi^*}$ and $j' < i < j$, $\omega_{j'} \succ_{\rho(i)} \omega_j = a_{\rho(i)}$, which contradicts $\{\rho(i), \rho(j)\}$ weakly blocking $\Psi(\succsim, \mathbf{K})$ via a .

²⁵Moreover, if for each $i \in N$, $\#\Omega_i = 1$, and $\mathbf{K} = (K_1, \dots, K_n) = (\omega_1, \dots, \omega_n)$, $\mathcal{GM}_n^\sigma(\succsim, \mathbf{K}) = \mathcal{M}_n(\succsim)$.

Case ii) $i \leq j'$. Since $i < j$ and $\omega_j \in D(\succsim_{\rho(i)})$ and $\omega_j \succ_{\rho(i)} \omega_{j'} = \Psi_{\rho(i)}(\succsim, \mathbf{K})$, by the definition of Ψ , either $\omega_{\rho(i)} \notin D(\succsim_{\rho(j)})$ or there is i' with $i' < i$, such that $\Psi_{\rho(j)}(\succsim, \mathbf{K}) = \omega_{i'}$. In either case, $\Psi_{\rho(j)}(\succsim, \mathbf{K}) \succ_{\rho(j)} \omega_i$, which contradicts $\{\rho(i), \rho(j)\}$ weakly blocking $\Psi(\succsim, \mathbf{K})$ via a .

Since both cases are exhaustive, $\Psi(\succsim, \mathbf{K}) \in \mathcal{C}(\succsim, \mathbf{K})$, and by the arguments in the previous paragraph, $\mathcal{C}(\succsim, \mathbf{K}) = \{\Psi(\succsim, \mathbf{K})\}$. \square

With Proposition 4 at hand, we characterize the generalized age based priority rule Ψ as the unique generalized pairwise exchange rule that satisfies *individual rationality*, *2-efficiency*, and *strategy-proofness* in D^{Π^*} .

Theorem 12. *A generalized rule $\Phi : \mathcal{D}^{\Pi^*} \times \mathcal{K} \rightarrow A^2$ satisfies individual rationality, 2-efficiency and strategy-proofness if and only if Φ is the generalized age based priority rule Ψ .*

Proof. The generalized age based priority rule Ψ satisfies *individual rationality* by its very definition of Ψ . Assume that Ψ does not satisfy *2-efficiency*, then there are $\succsim \in \mathcal{D}^{\Pi}$ and $a \in \mathcal{A}^2$ such that for each $i \in N$, $a_i \succsim_i \Psi_i(\succsim, \mathbf{K})$, and for some $i' \in N$, $a_{i'} \succ_{i'} \Psi_{i'}(\succsim, \mathbf{K})$. Let k be such that $a_{\rho(k)} \succ_{\rho(k)} \Psi_{\rho(k)}(\succsim, \mathbf{K})$. Note that $a_{\rho(k)} = \omega_{k'} \neq \bar{\omega}(\rho(k), \succsim_{\rho(k)})$, and $a_{\rho(k')} \neq \Psi_{\rho(k')}(\succsim, \mathbf{K})$, and since $a_i \succsim_i \Psi_i(\succsim, \mathbf{K})$ and preferences over desirable kidneys are strict, $a_{\rho(k')} \succ_{\rho(k')} \Psi_{\rho(k')}(\succsim, \mathbf{K})$. Thus, $\{\rho(i), \rho(j)\}$ weakly block $\Psi(\succsim, \mathbf{K})$ at \succsim via a , which contradicts Proposition 4, since $\Psi(\succsim, \mathbf{K}) \in \mathcal{C}(\succsim, \mathbf{K})$. We conclude showing that Ψ satisfies *strategy-proofness*. Let $i \in N$, $\succsim \in \mathcal{D}^{\Pi^*}$ and $\succsim'_i \in \mathcal{D}_i^{\Pi^*}$. Assume first that, $\psi_i^*(\succsim) = \psi_i^*(\succsim'_i, \succsim_{-i})$, then $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$. Assume now that $\psi_i^*(\succsim) \neq \psi_i^*(\succsim'_i, \succsim_{-i})$. Let $j, j' \in \mathbb{N}$ be such that $\Psi_i(\succsim) = \omega_j$ and $\Psi(\succsim'_i, \succsim_{-i}) = \omega_{j'}$. There are two cases:

Case i). $\omega_j = \bar{\omega}(i, \succsim_i)$. By the definition of Ψ , $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, and $\Psi_i(\succsim) \succsim_i \Psi_i(\succsim'_i, \succsim_{-i})$.

Case ii). $\omega_j \neq \bar{\omega}(i, \succsim_i)$. By the definition of $\Psi(\succsim)$, $\omega_j \in D(\succsim_i)$ and therefore $\omega_j \succ_i \bar{\omega}(i, \succsim_i)$. Since $\Psi_i(\succsim) \neq \Psi_i(\succsim'_i, \succsim_{-i})$, then either $j' < j$ with $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, or $j < j'$, or $\omega_{j'} \in \Omega_i$. In either case $\Psi_i(\succsim) \succsim_i \Psi_i(\succsim'_i, \succsim_{-i})$.

Since both cases are exhaustive, ψ^* satisfies *strategy-proofness*.

The proof of necessity follows from the same arguments of the proof of Theorem 2 in the text, since the multiple donors framework satisfies the conditions of Theorem 1 in Sönmez [36]. \square

Next we check that generalized age based priority rules satisfy the properties we are interested in for arbitrary age structures.

Proposition 5. *Let Π be an arbitrary age structure. The generalized age based priority rule $\Psi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow A^2$ satisfies individual rationality, 2-efficiency and strategy-proofness.*

Proof. By definition, Ψ satisfies *individual rationality*.

2-efficiency. Let $\succsim \in \mathcal{D}^\Pi$ and $\mathbf{K} \in \mathcal{K}$. Assume to the contrary that Ψ does not satisfy 2-efficiency. Then there is $a \in \mathcal{A}^2(\mathbf{K})$ such that $a_i \succsim_i \Psi_i(\succsim, \mathbf{K})$ for each $i \in N$ and there is j such that $a_j \succ_j \Psi_j(\succsim, \mathbf{K})$. Note that since Ψ satisfies *individual rationality*, $a \in \mathcal{I}(\succsim, \mathbf{K})$. Let $k \leq n'$ be such that $\rho(k) = j$ and such that for each $\mathcal{GM}_k(\succsim, \mathbf{K}) \subseteq \mathbf{S}_k(\succsim, \mathbf{K})$. That is, $\Psi_j(\succsim, \mathbf{K}) \in \mathcal{S}_k(\succsim, \mathbf{K})$ and there is no $b \in \mathcal{GM}_{k-1}(\succsim, \mathbf{K})$ such that $b_j \succ_j \Psi_j(\succsim, \mathbf{K})$. Hence $a \notin \mathcal{GM}_{k-1}(\succsim, \mathbf{K})$, and there is j' and $k' \leq k$ such that $\rho(k') = j$ and for each $a' \in \mathcal{GM}_{k-1}(\succsim, \mathbf{K})$, $a'_{j'} \succ_{j'} a_{j'}$. Since $\Psi(\succsim, \mathbf{K}) \in \mathcal{GM}_{k-1}(\succsim, \mathbf{K})$, $\Psi_{j'}(\succsim, \mathbf{K}) \succ_{j'} a_{j'}$, a contradiction.

Strategy-proofness. Let $i \in N$, $\succsim \in \mathcal{D}^{\Pi^*}$ and $\succsim'_i \in \mathcal{D}_i^{\Pi^*}$. Assume first that, $\psi_i^*(\succsim) = \psi_i^*(\succsim'_i, \succsim_{-i})$, then $\psi_i^*(\succsim) \succsim_i \psi_i^*(\succsim'_i, \succsim_{-i})$. Assume now that $\psi_i^*(\succsim) \neq \psi_i^*(\succsim'_i, \succsim_{-i})$. Let $j, j' \in \mathbb{N}$ be such that $\Psi_i(\succsim) = \omega_j$ and $\Psi(\succsim'_i, \succsim_{-i}) = \omega_{j'}$. There are two cases:

Case i). $\omega_j = \bar{\omega}(i, \succsim_i)$. By the definition of Ψ , $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, and $\Psi_i(\succsim) \succsim_i \Psi_i(\succsim'_i, \succsim_{-i})$.

Case ii). $\omega_j \neq \bar{\omega}(i, \succsim_i)$. Let l, l' be such that $j \in \Pi(l)$, $j' \in \Pi(l')$. By the definition of $\Psi(\succsim)$, $\omega_j \in D(\succsim_i)$ and therefore $\omega_j \succ_i \bar{\omega}(i, \succsim_i)$. Since $\Psi_i(\succsim) \neq \Psi_i(\succsim'_i, \succsim_{-i})$, then either $l' < l$ and $\omega_{j'} \in D(\succsim'_i) \setminus D(\succsim_i)$, or $l \leq l'$, or $\omega_{j'} \in \Omega_i$. In either case $\Psi_i(\succsim) \succsim_i \Psi_i(\succsim'_i, \succsim_{-i})$.

Since both cases are exhaustive, ψ^* satisfies *strategy-proofness*.

□

Since the set of donors of each patient may be private information, we introduce a non-manipulability property that takes into account patients' incentives to manipulate the PKE outcome by withholding their potential donors as well as misreporting their preferences.

Example 4. Consider N , \succsim , and \mathbf{K} as defined in Example 3. Let $K'_j = \Omega_j$ for $j \in \{1, 2\}$ and $K'_3 = \{\omega_5\}$. Note that

$$\Phi(\succsim, \mathbf{K}) = (\omega_3, \omega_2, \omega_4) \quad \text{and} \quad \Phi(\succsim, \mathbf{K}') = (\omega_1, \omega_5, \omega_2).$$

Hence, $\Phi_3(\succsim, \mathbf{K}') \succ_3 \Phi_3(\succsim, \mathbf{K})$.

The next theorem, shows that the generalized age based priority rule Ψ is immune to manipulations of groups of patients by withholding part of their donors.

Extended Weakly Coalitional Strategy-Proofness (EWCSPP). There are no $T \subseteq N$, $\succsim \in \mathcal{D}$, $\succsim'_T \in \mathcal{D}_T$, and $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$, such that for each $i \in T$,

$$\Phi_i(\succsim'_T, \succsim_{N \setminus T}, (\mathbf{K}'_T, \Omega_{N \setminus T})) \succ_i \Phi_i(\succsim, \Omega).$$

Theorem 13. The generalized age-based priority rule $\Psi : \mathcal{D}^\Pi \times \mathcal{K} \rightarrow \mathcal{A}^2$ satisfies EWCSPP.

Proof. For each $\succsim \in \mathcal{D}^\Pi$, and each $\mathbf{K}' \in \mathcal{K}$ such that for each patient i , $K'_i \subset \Omega_i$, because $\mathcal{I}(\succsim, \mathbf{K}') \subseteq \mathcal{I}(\succsim, \Omega)$, by the iterative definition of Ψ , for every patient i , $\Psi_i(\succsim, \Omega) \succsim_i \Psi_i(\succsim, \mathbf{K}')$. Then, for each $T \subseteq N$, each $\succsim \in \mathcal{D}^\Pi$, each $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$, for each $i \in T$:

$$\Psi_i(\succsim, \Omega) \succsim_i \Psi_i(\succsim, (\Omega_{N \setminus T}, \mathbf{K}'_T)). \quad (1)$$

Let $\mathbf{K} \in \mathcal{K}$. We next show that for each $\succsim \in \mathcal{D}^\Pi$ and each $T \subseteq N$, there is no $\succsim'_T \in \mathcal{D}_T^\Pi$ such that for each $i \in T$, $\Psi_i(\succsim'_T, \succsim_{N \setminus T}) \succ_i \Psi_i(\succsim)$. Assume to the contrary that there exist $\succsim \in \mathcal{D}^\Pi$, $T \subseteq N$, and $\succsim'_T \in \times_{i \in T} \mathcal{D}_i^\Pi$ such that for each $i \in T$, $\Psi_i(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \succ_i \Psi_i(\succsim, \mathbf{K})$.

Let kidney $j \leq n'$ be such that for each $\rho(j) \in T$, and for each $k < j$, either $\rho(k) \notin T$ or for no $i' \in N$ $\Psi_{i'}(\succsim) = \omega_k$. Since $\rho(j) \in T$, $\succsim_{\rho(j)} \neq \succsim'_{\rho(j)}$ and $\Psi_{\rho(j)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \succ_{\rho(j)} \Psi_{\rho(j)}(\succsim, \mathbf{K})$, necessarily $\Psi_{\rho(j)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \neq \Psi_{\rho(j)}(\succsim, \mathbf{K})$ and $\Psi_{\rho(j)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \in D(\succsim_{\rho(j)})$. There are two possibilities:

Case i). For each $i < j$, $\omega_i = \Psi_{i'}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$ if and only if $\omega_i = \Psi_{i'}(\succsim, \mathbf{K})$.

Since Ψ satisfies *individual rationality* and $\Psi_{\rho(j)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \in D(\succsim_{\rho(j)})$, there is $\omega_{j'} \notin \Omega_{\rho(j)}$ such that $\omega_{j'} = \Psi_{\rho(j)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$. By the definition of Ψ , $\omega_{\rho(j)} \in D(\succsim'_{\rho(j')}) \setminus D(\succsim_{\rho(j')})$. Because $\succsim'_{\rho(j')} \neq \succsim_{\rho(j')}$, $\rho(j') \in T$. However, since Ψ satisfies *individual rationality*, $\Psi_{\rho(j')}(\succsim, \mathbf{K}) \succsim_{\rho(j')} \omega_{\rho(j')} \succ_{\rho(j')} \omega_{\rho(j)} = \Psi_{\rho(j')}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$, which contradicts $\rho(j') \in T$.

Case ii). There exists $i < j$ such that either $\omega_i = \Psi_{i'}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \neq \Psi_{i'}(\succsim, \mathbf{K})$ or $\omega_i = \Psi_{i'}(\succsim, \mathbf{K}) \neq \Psi_{i'}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$. Let i^* be the smallest such i . Note that by the definition of T and j , $i^* \notin T$. If $\Psi_{\rho(i^*)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \succ_{\rho(i^*)} \Psi_{\rho(i^*)}(\succsim, \mathbf{K})$, by the definition of generalized age based priority rule, there exists $k \in T$ such that $\omega_{i'} \notin D(\succsim_k)$ and $\omega_{i'} \in D(\succsim'_k)$ and $\Psi_{\rho(i^*)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}) \in \Omega_k$. However, by *individual rationality*, $\Psi_k(\succsim, \mathbf{K}) \succ_k \omega_{i'} = \Psi_k(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$, which contradicts $k \in T$. Finally, if $\Psi_{\rho(i^*)}(\succsim, \mathbf{K}) \succ_{\rho(i^*)} \Psi_{\rho(i^*)}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$, then there exists $k' \in T$ such that $\Psi_{\rho(i^*)}(\succsim) \in \Omega_{k'}$ and $\omega_{\rho(i^*)} \in D(\succsim_{k'}) \setminus D(\succsim'_{k'})$, and by the definition of Ψ and i^* , $\omega_{i^*} = \Psi_{k'}(\succsim, \mathbf{K}) \succ_{k'} \Psi_{k'}(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K})$, which contradicts $k' \in T$.

Repeating the arguments in the previous paragraphs, we obtain that there are no $T \subset N$, $\succsim \in \mathcal{D}^\Pi$, $\succsim'_T \in \mathcal{D}^\Pi_T$ and $K' \in \mathcal{K}$, such that for each $i \in T$:

$$\Psi_i(\succsim'_T, \succsim_{N \setminus T}, \mathbf{K}') \succ_i \Psi_i(\succsim, \mathbf{K}'). \quad (2)$$

Combining equations (1) and (2), and letting $\mathbf{K}' = (\Omega_{N \setminus T}, \mathbf{K}'_T)$, we obtain that there are no $T \subset N$, $\succsim \in \mathcal{D}^\Pi$, $\succsim'_T \in \mathcal{D}^\Pi_T$, and $\mathbf{K}'_T \in \times_{i \in T} \mathcal{K}_i$ such that for each $i \in T$

$$\Psi_i(\succsim'_T, \succsim_{N \setminus T}, (\Omega_{N \setminus T}, \mathbf{K}'_T)) \succ_i \Psi_i(\succsim, \Omega).$$

□

Before concluding this appendix, it is worth pointing out that our rules are immune to other form of misrepresentation of the information about the set of donors apart from withholding some potential donors.

Remark 2. *Note that generalized age based priority rules are also immune to manipulation by the introduction of dummy donors. That is, a patient does not improve by presenting a donor whose kidneys are not compatible with any other patient. That is, consider patient i and let $\Omega_i = \bar{\Omega}_i \cup \{\omega\}$. Let $\succsim \in \mathcal{D}^\Pi$ be such that for each patient $j \neq i$, $\omega \notin D(\succsim_j)$. Then, $\Psi(\succsim, \Omega) = \Psi(\succsim, (\Omega_{N \setminus \{i\}}, \bar{\Omega}_i))$.*